

Regulators and cycle maps in higher-dimensional differential algebraic K -theory

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Abstract

We develop differential algebraic K -theory of regular and separated schemes of finite type over $\mathbf{Spec}(\mathbb{Z})$. Our approach is based on a new construction of a functorial, spectrum level Beilinson regulator using differential forms. We construct a cycle map which represents differential algebraic K -theory classes by geometric vector bundles. As an application we derive Lott's relation between short exact sequences of geometric bundles with a higher analytic torsion form.

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1 Introduction

1.1 Differential algebraic K -theory

The algebraic K -theory of a number ring or, more generally, of a regular and separated scheme of finite type over $\mathbf{Spec}(\mathbb{Z})$ is an interesting and complicated object. Rationally, the algebraic K -groups are studied using natural transformations to cohomology which are called regulators. In the case of number rings these regulators lead to a complete calculation of the rational algebraic K -groups by Borel [Bor74]. In his seminal paper [Beĭ84] Beilinson formulated far reaching conjectures predicting the order of vanishing and the leading coefficient of the L -functions of (the generic fibre of) a (flat, proper) scheme as above in terms of regulators (up to rational multiples). In particular, they predict the dimensions of the rational K -groups in terms of the dimensions of the kernels and images of the regulators. We will give a precise statement of this part of Beilinson's conjectures in 3.5.

These conjectures were proven by Borel [Bor77, Bor74] for number rings. Beyond this special case they are currently – apart from partial results – open. Any further progress would require a deep understanding of regulators and algebraic K -theory, and a way of constructing K -theory classes and computing their images under the regulator maps.

In [BG] we started a new approach to understand aspects of the algebraic K -theory of a number ring R by embedding the cohomology theory represented by the algebraic K -theory spectrum of R into a differential cohomology theory. In general, a differential extension of a cohomology theory can be evaluated on smooth manifolds. It combines the homotopy theoretic information with differential form data in a non-trivial way leading to interesting secondary invariants. We refer to [BS10] for an axiomatic description of differential cohomology theories. A differential extension of a cohomology theory has the potential to receive refined characteristic classes which encode geometric data. As a classical example, the differential extension of integral cohomology receives refined Chern classes for vector bundles with connection [CS85].

The present paper introduces the differential algebraic K -theory $\widehat{\mathbf{K}}(X)^0$ for a regular and separated scheme X of finite type over $\mathbf{Spec}(\mathbb{Z})$. This is a functor which can be evaluated on smooth manifolds (the evaluation will be denoted $\widehat{\mathbf{K}}(X)^0(M)$). It may also be considered as an alternative version of higher arithmetic K -theory which was initiated by Gillet and Soulé [GS90] and then elaborated by Takeda [Tak05] building on work by Burgos and Wang [BW98]. A third approach to arithmetic K -theory is given by Scholbach in [Sch12] which uses the abstract framework of stable homotopy theory of schemes.

The present paper represents a first stage of the systematic development of differential algebraic K -theory. Its goals are to present the construction of differential algebraic K -theory for regular and separated schemes of finite type over $\mathbf{Spec}(\mathbb{Z})$ on the one hand, and the application of this theory to a proof of Lott's relation on the other hand. Since it combines information about K -theory classes and their images under the regulator maps one might hope that differential algebraic K -theory will be a useful tool for the construction of K -theory classes and the computation of regulators. Examples of this

type and other applications will be taken up in subsequent papers.

We now start, reviewing [BG], with a description of the framework around Lott's relation. In Subsection 1.3 we give a more detailed introduction to $\widehat{KR}(X)^0$.

1.2 Number rings and a review of [BG]

In the context of algebraic K -theory of a ring R the relevant topological object over a manifold M is a bundle V of finitely generated projective R -modules. It gives rise to a class

$$\mathrm{cycl}(V) \in KR^0(M) ,$$

where KR^* denotes the cohomology theory represented by the K -theory spectrum KR of R . A regulator $r : KR \rightarrow H\mathbb{R}[p]$, where HC or $H(C)$ denotes the Eilenberg-MacLane spectrum of the complex C (see A.5), defines a natural transformation between cohomology theories $r : KR^0(M) \rightarrow H^p(M; \mathbb{R})$. We consider the question how to calculate the class $r(\mathrm{cycl}(V)) \in H^p(M; \mathbb{R})$.

Let us assume that R is a number ring. We choose an embedding $\sigma : R \rightarrow \mathbb{C}$ and take $p = 2k + 1$ for $k \in \mathbb{N}$. These choices determine a regulator

$$r_{\sigma,p} : KR \rightarrow H\mathbb{R}[p]$$

first introduced by Borel [Bor74] which has the following description. The complexification $V_\sigma := V \otimes_{R,\sigma} \mathbb{C}$ is a complex vector bundle with a flat connection ∇^{V_σ} . If we choose a hermitean metric h^{V_σ} , then we can form the adjoint connection $\nabla^{V_\sigma,*}$. The p -component $\tilde{\mathbf{ch}}_p(\nabla^{V_\sigma,*}, \nabla^{V_\sigma})$ of the transgression of the Chern character form is closed. Up to normalization, this is the Kamber-Tondeur form of the flat bundle with metric $(V_\sigma, \nabla^{V_\sigma}, h^{V_\sigma})$ introduced in [BL95]. Its cohomology class is independent of the choice of the metric. The regulator is now characterized by

$$r_{\sigma,p}(\mathrm{cycl}(V)) = [\tilde{\mathbf{ch}}_p(\nabla^{V_\sigma,*}, \nabla^{V_\sigma})] \in H^p(M; \mathbb{R}) .$$

The choice of a conjugation invariant collection h^V of metrics h^{V_σ} for all embeddings $\sigma : R \rightarrow \mathbb{C}$ is called a geometry on V . One motivation for introducing the differential extension \widehat{KR}^0 of algebraic K -theory of R is that it can capture a refined cycle class

$$\hat{\mathrm{cycl}}(V, h^V) \in \widehat{KR}^0(M) \tag{1}$$

which combines the information about the class $\mathrm{cycl}(V) \in KR^0(M)$ and the characteristic forms $\tilde{\mathbf{ch}}_p(\nabla^{V_\sigma,*}, \nabla^{V_\sigma})$ with secondary invariants.

The successful realization of this program is one of the main results of [BG]. The cycle map provides the bridge from differential to arithmetic geometry. It allows to transfer constructions from manifold theory into the realm of algebra. In the following we discuss two examples, the transfer index conjecture and Lott's relation.

For any cohomology theory E^* represented by a spectrum E the Becker-Gottlieb transfer is a push-forward operation $\mathrm{tr} : E^*(M) \rightarrow E^*(B)$ along a proper submersion $\pi : M \rightarrow B$

of manifolds. It can be used to understand aspects of the fine-structure of the spectrum E using differential topology.

We apply this to the algebraic K -theory spectrum KR of a number ring R . A bundle of R -modules V can be pushed forward along π by taking fibre-wise cohomology. In sheaf theoretic terms, one considers the higher derived images $R^i\pi_*(V)$. The Becker-Gottlieb transfer and this push-forward of bundles of R -modules can be compared via the cycle map leading to the following equality in $KR^0(M)$:

$$\sum_{i \geq 0} (-1)^i \text{cycl}(R^i\pi_*(V)) = \text{tr}(\text{cycl}(V)) . \quad (2)$$

The Bismut-Lott index theorem [BL95] implies that (2) holds true in $H^p(B; \mathbb{R})$ after application of the regulator $r_{\sigma,p}$ described above. The equality (2) in $KR^0(M)$ itself is a consequence of the Dwyer-Weiss-Williams index theorem [DWW03].

The fundamental question considered in [BG] concerned the refinement of (2) to differential algebraic K -theory. First of all, in [BG] we construct a differential Becker-Gottlieb transfer

$$\hat{\text{tr}} : \hat{E}^*(M) \rightarrow \hat{E}^*(B)$$

for every differential cohomology theory \hat{E} . This differential refinement of tr depends on the additional choice of a Riemannian structure g^π on π .

If (V, h^V) is a bundle of R -modules with metric we define metrics $h_{L^2}^{R^i\pi_*(V)}$ on the bundles $R^i\pi_*V$ using fibre-wise Hodge theory. These constructions allow to lift both sides of (2) to differential algebraic K -theory. A simple check using the local version of the Bismut-Lott index theorem [BL95] shows that the naive differential version of (2) is not true. But the theory of Bismut-Lott provides a natural candidate for a correction term which can be expressed in terms of a higher analytic torsion form $\mathcal{T}(\pi, g^\pi, V, h^V)$. We refer to [BG] for details. Quite generally differential cohomology receives a map from differential forms which is denoted by the symbol a as it appears in the following formula.

Conjecture 1.1. *The transfer index conjecture (TIC) predicts that*

$$\sum_{i \geq 0} (-1)^i \hat{\text{cycl}}(R^i\pi_*(V), h_{L^2}^{R^i\pi_*(V)}) + a(\mathcal{T}(\pi, g^\pi, V, h^V)) = \hat{\text{tr}}(\hat{\text{cycl}}(V, h^V)) \quad (3)$$

holds true in $\widehat{KR}^0(B)$.

The TIC is an interdisciplinary, still unproven, statement which combines homotopy theory, global analysis and arithmetic. It subsumes known results like the local version of the Bismut-Lott index theorem [BL95], the Dwyer-Weiss-Williams index theorem [DWW03], or a version of the Cheeger-Müller theorem [Che79], [Mül78]. Further special cases and arithmetic consequences are discussed in [BG, Sec. 5].

We now describe Lott's relation. We consider a short exact sequence

$$\mathcal{V} : 0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0, \quad g^\mathcal{V} := (g^{V_i})_{i=0,1,2}, \quad (4)$$

of bundles over M of finitely generated projective modules over a number ring R together with a choice of geometries. In $KR^0(M)$ we then have the relation

$$\text{cycl}(V_0) - \text{cycl}(V_1) + \text{cycl}(V_2) = 0 . \quad (5)$$

It is again a natural question whether this equality refines to differential algebraic K -theory. Again by the theory of Bismut-Lott [BL95], the naive refinement does not hold in general, but there is a natural correction term given by a finite-dimensional version of the higher-analytic torsion form $\mathcal{T}(\mathcal{V}, g^\mathcal{V})$. In [BG] we conjectured and partially verified the following result:

Theorem 1.2. *Lott's relation*

$$\widehat{\text{cycl}}(V_0, g^{V_0}) - \widehat{\text{cycl}}(V_1, g^{V_1}) + \widehat{\text{cycl}}(V_2, g^{V_2}) = a(\mathcal{T}(\mathcal{V}, g^\mathcal{V}))$$

holds true in $\widehat{KR}^0(B)$.

We call this Lott's relation since a version of it was taken by Lott in [Lot00] as the starting point for the construction of a geometric analogue of the flat part of $\widehat{KR}^0(M)$. We refer to [BG] for further details and consequences of Lott's relation. One of the main achievements of the present paper is a complete proof of Lott's relation 1.2, see Theorem 5.16.

1.3 The higher-dimensional case and the main results of this paper

The theory of [BG] applies to the differential algebraic K -theory of number rings. The goal of the present paper is to set-up the first step of a generalization of the theory from number rings to general regular and separated schemes X of finite type over $\text{Spec}(\mathbb{Z})$.

In order to define a differential extension $\widehat{\mathbf{K}}(X)$ of the cohomology theory determined by the K -theory spectrum $\mathbf{K}_\mathbb{Z}(X)$ one must fix a suitable choice of differential form data. In the present paper we work with the absolute Hodge complex as suggested by Beilinson's conjectures (cf. [Beĭ86]). It will be denoted by $\mathbf{DR}_\mathbb{Z}(X)$. One difference to the standard choice in differential cohomology theory is that $\mathbf{DR}_\mathbb{Z}(X)$ does not exactly model the real homotopy type of $\mathbf{K}_\mathbb{Z}(X)$, and even if X is proper the precise relation between its cohomology and the homotopy groups $\pi_*(\mathbf{K}_\mathbb{Z}(X))$ is only conjectural.

In general, a differential extension of a cohomology theory E can be considered as a byproduct of a geometric construction of the homotopy fibre of the realification map $E \rightarrow E\mathbb{R}$. In the present situation of algebraic K -theory $\mathbf{K}_\mathbb{Z}(X)$ the realification is replaced by the Beilinson regulator $\mathbf{r}_\mathbb{Z}^{\text{Beil}} : \mathbf{K}_\mathbb{Z}(X) \rightarrow H(\mathbf{DR}_\mathbb{Z}(X))$. For a smooth manifold M we have an exact sequence (see (64))

$$\mathbf{K}_\mathbb{Z}(X)^{-1}(M) \xrightarrow{\mathbf{r}_\mathbb{Z}^{\text{Beil}}} H(\mathbf{DR}_\mathbb{Z}(X))^{-1}(M) \xrightarrow{a} \widehat{\mathbf{K}}(X)^0(M) \xrightarrow{I} \mathbf{K}_\mathbb{Z}(X)^0(M) \rightarrow 0$$

and a map

$$R : \widehat{\mathbf{K}}(X)^0(M) \rightarrow Z^0(\mathbf{DR}_{Mf, \mathbb{Z}}(M \times X)) .$$

A differential algebraic K -theory class $u \in \widehat{\mathbf{K}}(X)^0(M)$ contains the information about an underlying topological class and a differential form representing the regulator of the former. These pieces are extracted by the maps I and R . The flat part (Definition 5.2)

$$\widehat{\mathbf{K}}(X)_{flat}^0(M) := \ker \left(R : \widehat{\mathbf{K}}(X)^0(M) \rightarrow Z^0(\mathbf{DR}_{Mf, \mathbb{Z}}(M \times X)) \right)$$

fits into the exact sequence (see (69))

$$\mathbf{K}_{\mathbb{Z}}(X)^{-1}(M) \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}} H(\mathbf{DR}_{\mathbb{Z}}(X))^{-1}(M) \xrightarrow{a} \widehat{\mathbf{K}}(X)_{flat}^0(M) \xrightarrow{I} \mathbf{K}_{\mathbb{Z}}(X)^0(M) \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}} H(\mathbf{DR}_{\mathbb{Z}}(X))^0(M) .$$

We now explain the relation to previous constructions of arithmetic algebraic K -theory. Let $j : \text{pt} \rightarrow S^n$ be the inclusion of the north pole. Then we have isomorphisms

$$\mathbf{K}_n(X) \cong \ker (j^* : \mathbf{K}_{\mathbb{Z}}(X)^0(S^n) \rightarrow \mathbf{K}_{\mathbb{Z}}(X)^0(\text{pt}))$$

and

$$H^{-n}(\mathbf{DR}_{\mathbb{Z}}(X)) \cong \ker (j^* : H(\mathbf{DR}_{\mathbb{Z}}(X))^0(S^n) \rightarrow H(\mathbf{DR}_{\mathbb{Z}}(X))^0(\text{pt})) .$$

The relation of this cohomology with absolute Hodge cohomology of X is given by

$$H^*(\mathbf{DR}_{\mathbb{Z}}(X)) \cong \bigoplus_{p \geq 0} H_{\text{Hodge}}^*(X, \mathbb{R}(p)[2p]) ,$$

see (25). If we set

$$\overline{\mathbf{K}}_n(X) := \ker (j^* : \widehat{\mathbf{K}}(X)_{flat}^0(S^n) \rightarrow \widehat{\mathbf{K}}(X)_{flat}^0(\text{pt})) ,$$

then this group fits into an exact sequence

$$\mathbf{K}_{n+1}(X) \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}} H^{-n-1}(\mathbf{DR}_{\mathbb{Z}}(X)) \xrightarrow{a} \overline{\mathbf{K}}_n(X) \xrightarrow{I} \mathbf{K}_n(X) \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}} H^{-n}(\mathbf{DR}_{\mathbb{Z}}(X)) . \quad (6)$$

The group $\overline{\mathbf{K}}_n(X)$ can thus be understood as a version of the higher arithmetic algebraic K -theory¹ according to [Sou92, III.2.3.4], [Del87].

Based on the differential form level construction of the Beilinson regulator by Burgos and Wang [BW98], Takeda defines in [Tak05] a group $\widehat{\mathbf{K}}_n^{\text{Takeda}}(X)$ for proper X together with a characteristic form map \mathbf{ch} . The group $\widehat{\mathbf{K}}_n^{\text{Takeda}}(X)$ is the analogue of our

$$\widehat{\mathbf{K}}_n^{BT}(X) := \ker (j^* : \widehat{\mathbf{K}}(X)^0(S^n) \rightarrow \widehat{\mathbf{K}}(X)^0(\text{pt})) .$$

It is not the same because of a different choice of differential form data computing absolute Hodge cohomology. Takeda's version of arithmetic algebraic K -theory is

$$\overline{\mathbf{K}}_n^{\text{Takeda}}(X) := \ker(\mathbf{ch}) .$$

¹The usual notation for arithmetic algebraic K -theory is $\widehat{\mathbf{K}}_n(X)$. In the present paper we use $\overline{\mathbf{K}}_n(X)$ in order to avoid a confusing inflation of hatted symbols.

It fits into an exact sequence which is the analogue of (6). We expect that there is a natural isomorphism

$$\overline{\mathbf{K}}_n(X) \cong \overline{\mathbf{K}}_n^{\text{Takeda}}(X) .$$

We now mention the main achievements of the present paper:

1. **The construction of $\widehat{\mathbf{K}}(X)^0$ (Definition 5.1):** The construction of the differential algebraic K -theory functor $\widehat{\mathbf{K}}(X)^0$ requires the development of a framework that enables one to understand the Beilinson regulator on the differential form level. This occupies the first two and part of the fourth section.
2. **The cycle map (Subsection 5.3):** In the setup of [BG] the construction of the cycle map (1) was a considerable task. The contribution of the present paper is a higher-dimensional generalization together with a great simplification. Let M be a smooth manifold and consider a sheaf V of locally free, finitely generated $\mathbf{pr}_X^* \mathcal{O}_X$ -modules over $M \times X$. We introduce the notion of a geometry g^V on V which allows us to define a characteristic form $\omega(g^V) \in Z^0(\mathbf{DR}_{Mf,\mathbb{Z}}(M \times X))$, see Definition 4.13. These characteristic forms lead to a very functorial construction of the Beilinson regulator maps in Definition 4.22. The cycle map associates in a natural way a differential algebraic K -theory class

$$\widehat{\text{cyl}}(V, g^V) \in \widehat{\mathbf{K}}(X)^0(M)$$

to a geometric bundle (V, g^V) .

3. **The proof of Lott's relation (Theorem 5.16):** Most interestingly, this is a mere consequence of the fact that the calculus developed in [BG] for number rings has a higher-dimensional generalization.

As a simple consequence of the way the regulators are constructed we rederive the generalization of the Karoubi regulator first obtained in [Tam12, Section 3.5].

Besides the above mentioned results, we think that the formalism and techniques developed in the present paper should be of independent interest.

We now describe the contents of the various sections.

In Section 2 we define algebraic K -theory in terms of an ∞ -categorical version of commutative group completion. The main purpose is to analyze the construction of regulator maps and to reduce the amount of data to prescribe and check to a practical level. We give a precise way to produce ∞ -categorical objects (e.g. the regulators) from classical data (e.g. characteristic forms).

In Section 3 we give a construction of the Beilinson regulator using the general method developed in Section 2. Here we use some of the differential geometry which will be set up in Section 4. The application to the Karoubi regulator is discussed in Subsection 3.6. In Section 4 we develop the differential geometry of locally free, finitely generated sheaves of $\mathbf{pr}_X^* \mathcal{O}_X$ -modules V on spaces of the form $M \times X$. We introduce the corresponding

absolute Hodge complexes $\mathbf{DR}_{Mf,\mathbb{Z}}$, define the notion of a geometry g^V on V , and we construct the characteristic forms $\omega(g^V) \in Z^0(\mathbf{DR}_X(M))$. At the end, in Subsection 5, we give the Definition 5.1 of the differential algebraic K -theory functor

$$M \times X \mapsto \widehat{\mathbf{K}}^0(M \times X)$$

and derive its immediate properties. In Subsection 5.3 we construct the cycle map

$$\widehat{\text{cyc1}}: \pi_0(i\text{Vect}_{Mf,\mathbb{Z}}^{\text{geom}}(M \times X)) \ni (V, g^V) \mapsto \widehat{\text{cyc1}}(V, g^V) \in \widehat{\mathbf{K}}^0(M \times X) .$$

In Section 5.4 we give the precise relationship between the specialization of the theory of the present work to number rings R and the set-up of [BG]. The main result is the verification of Lott's relation (Theorem 1.2) in Theorem 5.16.

We have collected various notations, conventions, and technicalities in Appendix A and will refer there from time to time in the main text.

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2 Regulators

2.1 Definition of algebraic K -theory

In this subsection we describe the algebraic K -theory spectrum of a symmetric monoidal category in terms of nerves and group completion. This will later be applied to the categories of finitely generated projective modules over a ring or vector bundles on a regular scheme.

We start with the inclusion of commutative monoids into commutative groups which is part of an adjunction

$$K_0 : \mathbf{CommMon}(\mathbf{Set}) \rightleftarrows \mathbf{CommGroup}(\mathbf{Set})$$

whose left-adjoint is the group completion functor K_0 , also called the Grothendieck construction.

The ∞ -category $\mathbf{N}(\mathbf{sSet})[W^{-1}]$ of simplicial sets (with weak equivalences inverted, cf. A.2) has a symmetric monoidal structure given by the cartesian product. This allows to extend the notions of commutative monoids and groups to this context. Moreover, the notion of group completion generalizes. In fact, the forgetful map from commutative groups to commutative monoids in $\mathbf{N}(\mathbf{sSet})[W^{-1}]$ is again the right-adjoint of an adjunction

$$\Omega B : \mathbf{CommMon}(\mathbf{N}(\mathbf{sSet})[W^{-1}]) \rightleftarrows \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}]) ,$$

see Subsection A.3 We use the symbol ΩB resembling an explicit model for the group completion since we want to reserve the letter K to denote the K -theory spectrum functor. The infinite loop space functor provides an equivalence

$$\Omega^\infty : \mathbf{N}(\mathbf{Sp}^{\geq 0})[W^{-1}] \xrightarrow{\sim} \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}])$$

between the ∞ -categories of connective spectra and commutative groups in simplicial sets. The composition of its inverse with the forgetful functor from connective to all spectra will be denoted by

$$\mathbf{sp} : \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}]) \rightarrow \mathbf{N}(\mathbf{Sp})[W^{-1}] . \quad (7)$$

One source of commutative monoids in simplicial sets are nerves of symmetric monoidal categories. In fact, the nerve functor $\mathbf{N} : \mathbf{N}(\mathbf{Cat})[W^{-1}] \rightarrow \mathbf{N}(\mathbf{sSet})[W^{-1}]$ is symmetric monoidal with respect to the cartesian structures and therefore induces a transformation

$$\mathbf{N} : \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]) \rightarrow \mathbf{CommMon}(\mathbf{N}(\mathbf{sSet})[W^{-1}]) .$$

Moreover, we have a functor $i : \mathbf{Cat} \rightarrow \mathbf{Cat}$ which maps any category \mathcal{C} to its subcategory $i\mathcal{C}$ of isomorphisms. The induced functor $i : \mathbf{N}(\mathbf{Cat})[W^{-1}] \rightarrow \mathbf{N}(\mathbf{Cat})[W^{-1}]$ is also symmetric monoidal.

Definition 2.1. *We call the composition*

$$K := \mathbf{sp} \circ \Omega B \circ \mathbf{N} \circ i : \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]) \rightarrow \mathbf{N}(\mathbf{Sp})[W^{-1}]$$

the algebraic K-theory functor.

The considerations above generalize to diagrams indexed by a simplicial set \mathbf{S} . In order to simplify the notation we retain the symbol of a functor in order to denote its object wise extension to diagrams. We can make the following definition

Definition 2.2. *The connective algebraic K-theory of a diagram*

$$\mathcal{V} \in \mathbf{Fun}(\mathbf{S}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]))$$

of symmetric monoidal categories is the diagram of spectra

$$K(\mathcal{V}) \in \mathbf{Fun}(\mathbf{S}, \mathbf{N}(\mathbf{Sp})[W^{-1}]) , \quad K(\mathcal{V}) := \mathbf{sp}(\Omega B(\mathbf{N}(i\mathcal{V}))) .$$

2.2 The space of regulators

Let \mathcal{V} be a symmetric monoidal category. To determine the homotopy type of $K(\mathcal{V})$ is in general a difficult task. So we are interested, as a first approximation, in some cohomological information about $K(\mathcal{V})$. We consider cohomology theories represented by Eilenberg-MacLane spectra. The basic Eilenberg-MacLane spectrum is $H\mathbb{Z}$ whose only non-trivial homotopy group is $\pi_0(H\mathbb{Z}) \cong \mathbb{Z}$. It is a commutative ring spectrum and therefore gives rise to an ∞ -category of $H\mathbb{Z}$ -module spectra $\mathbf{Mod}(H\mathbb{Z})$. This ∞ -category turns out to be equivalent to the ∞ -category of chain complexes $\mathbf{N}(\mathbf{Ch})[W^{-1}]$. The equivalence

$$H : \mathbf{N}(\mathbf{Ch})[W^{-1}] \xrightarrow{\sim} \mathbf{Mod}(H\mathbb{Z})$$

is called the Eilenberg-MacLane equivalence. For more details we refer to Subsection A.5.

We have a forgetful functor

$$\mathbf{Mod}(H\mathbb{Z}) \rightarrow \mathbf{N}(\mathbf{Sp})[W^{-1}]$$

which will often be used without further notice in order to consider an $H\mathbb{Z}$ -module spectrum just as a spectrum.

Given a symmetric monoidal category $\mathcal{V} \in \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}])$ and a chain complex $F \in \mathbf{N}(\mathbf{Ch})[W^{-1}]$ we can now consider the commutative group

$$\Omega^\infty \mathbf{Map}(K(\mathcal{V}), H(F)) \in \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}]) ,$$

where $\mathbf{Map}(K(\mathcal{V}), H(F)) \in \mathbf{N}(\mathbf{Sp})[W^{-1}]$ denotes the mapping spectrum. The homotopy groups of $\Omega^\infty \mathbf{Map}(K(\mathcal{V}), H(F))$ are cohomology groups of $K(\mathcal{V})$ with coefficients in F . For example

$$\pi_0(\Omega^\infty \mathbf{Map}(K(\mathcal{V}), H(\mathbb{R}[-i]))) \cong H^i(K(\mathcal{V}); \mathbb{R}) .$$

In order to give a precise functorial definition of regulators we again extend these considerations to diagrams. We consider a diagram of chain complexes

$$F \in \mathbf{Fun}(\mathbf{S}, \mathbf{N}(\mathbf{Ch})[W^{-1}]) \tag{8}$$

and a diagram

$$\mathcal{V} \in \mathbf{Fun}(\mathbf{S}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]))$$

of symmetric monoidal categories. For two diagrams of spectra $K, L \in \mathbf{Fun}(\mathbf{S}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$ we let $\mathbf{Map}(K, L) \in \mathbf{N}(\mathbf{Sp})[W^{-1}]$ denote the mapping spectrum between the diagrams.

Definition 2.3. *The commutative group of F -valued regulators for \mathcal{V} is defined by*

$$\mathbf{Reg}(\mathcal{V}, F) := \Omega^\infty \mathbf{Map}(K(\mathcal{V}), H(F)) \in \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}]) .$$

Below we also use the term regulator for elements in $\pi_0(\mathbf{Reg}(\mathcal{V}, F))$.

2.3 Regulators and characteristic cocycles

In general, because of the ∞ -categorical nature of the objects involved, it appears to be a complicated task to produce regulators explicitly. Fortunately, the notion of a characteristic cocycle introduced below in Definition 2.4 amounts to a formidable reduction of complexity. The main goal of the present subsection is to indicate a general machine to produce regulators from characteristic cocycles.

In our applications \mathcal{V} is the symmetric monoidal category of certain vector bundles where the symmetric monoidal structure is implemented by the direct sum. Using differential geometric methods we construct additive characteristic forms. In this case it turns out to be a simple matter to produce characteristic cocycles from these characteristic forms. Plugging the resulting characteristic cocycle into the machine described here we get the desired regulators.

We consider a simplicial set \mathbf{S} , a diagram

$$\mathcal{V} \in \mathbf{Fun}(\mathbf{S}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]))$$

of symmetric monoidal categories and a diagram

$$F \in \mathbf{Fun}(\mathbf{S}, \mathbf{N}(\mathbf{Ch}))$$

of chain complexes. Note that we do not invert the quasi-isomorphisms here on purpose. We derive from \mathcal{V} the diagram of commutative monoids of isomorphism classes of objects of \mathcal{V}

$$\pi_0(i\mathcal{V}) \in \mathbf{Fun}(\mathbf{S}, \mathbf{N}(\mathbf{CommMon}(\mathbf{Set}))).$$

and from F the diagram of degree-zero cycles

$$Z^0(F) \in \mathbf{Fun}(\mathbf{S}, \mathbf{N}(\mathbf{Ab})) .$$

Definition 2.4. *A characteristic cocycle is a transformation*

$$\omega : \pi_0(i\mathcal{V}) \rightarrow Z^0(F)$$

between objects of $\mathbf{Fun}(\mathbf{S}, \mathbf{N}(\mathbf{CommMon}(\mathbf{Set})))$

We can consider sets as categories with only identity morphisms. Similarly, commutative monoids are symmetric monoidal categories. In this sense we have a symmetric monoidal functor $i\mathcal{V} \rightarrow \pi_0(i\mathcal{V})$ which associates to every object its isomorphism class. We consider the composition

$$i\mathcal{V} \rightarrow \pi_0(i\mathcal{V}) \xrightarrow{\omega} Z^0(F)$$

of morphisms in $\mathbf{Fun}(\mathbf{S}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]))$ to which we apply the algebraic K -theory functor. We get a map

$$K(\mathcal{V}) \xrightarrow{K(\omega)} K(Z^0(F)) . \quad (9)$$

We now use the commutativity of the following diagram (see (119))

$$\begin{array}{ccc} \mathbf{N}(\mathbf{Ab}) & \longrightarrow & \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]) \\ \downarrow S^0 & & \downarrow K \\ \mathbf{N}(\mathbf{Ch})[W^{-1}] & \xrightarrow{H} & \mathbf{N}(\mathbf{Sp})[W^{-1}] \end{array}$$

where the left vertical arrow maps an abelian group A to the chain complex $S^0(A)$ with the group A placed in degree zero, and the upper horizontal transformation interprets an abelian group as a symmetric monoidal category. This gives an equivalence

$$K(Z^0(F)) \xrightarrow{\sim} H(S^0(Z^0(F))) . \quad (10)$$

We have a natural map of diagrams of chain complexes

$$S^0(Z^0(F)) \rightarrow F$$

which induces

$$H(S^0(Z^0(F))) \rightarrow H(F) . \quad (11)$$

Definition 2.5. *We define the regulator*

$$r(\omega) \in \pi_0(\mathbf{Reg}(\mathcal{V}, F))$$

associated to the characteristic cocycle $\omega : \pi_0(i\mathcal{V}) \rightarrow Z^0(F)$ to be the composition

$$K(\mathcal{V}) \xrightarrow{(11) \circ (10) \circ (9)} H(F) .$$

At first sight, this definition of a regulator might seem naive since it is non-trivial only on $\pi_0(K(\mathcal{V}))$. Interesting information comes in when one evaluates this regulator on a diagram in \mathbf{S} . In our applications, for instance, this will be the cosimplicial object of standard simplices (see Subsection 4.6).

3 Beilinson's regulator

After having set up the general machinery to produce regulators in the previous section we now aim for the concrete case of regular, separated schemes of finite type over \mathbb{C} and then over \mathbb{Z} . First we define the relevant target for the regulator map. In view of Beilinson's famous conjectures [Bei84, Bei86] the appropriate cohomology is the absolute Hodge cohomology of the scheme considered. In order to construct a differential version of algebraic K -theory for such schemes we need a complex computing absolute Hodge cohomology built out of forms, with a good functorial behaviour. This is accomplished in Subsections 3.1 and 4.2 building upon work of Burgos [Bur94].

3.1 The absolute Hodge complex $\mathbf{DR}_{\mathbb{C}}$

By a variety over \mathbb{C} we will mean a separated scheme such that all connected components are of finite type over \mathbb{C} . We let $\mathbf{Sm}_{\mathbb{C}}$ denote the site of smooth varieties over \mathbb{C} with the topology given by Zariski open coverings. We further consider the ∞ -category

$$\mathbf{S}_{\mathbb{C}} := \mathbf{N}(\mathbf{Sm}_{\mathbb{C}}^{op}) .$$

Let X be a smooth variety over \mathbb{C} . According to Nagata [Nag62] and Hironaka [Hir64] there exists an open immersion $j : X \hookrightarrow \overline{X}$ into a smooth variety \overline{X} over \mathbb{C} such that each connected component of \overline{X} is proper over \mathbb{C} and each connected component of $D := \overline{X} - X$ is a divisor with normal crossings. We call \overline{X} a good compactification of X .

The set of \mathbb{C} -valued points $X(\mathbb{C})$ has a natural structure as a complex manifold. By abuse of notation we will denote it simply by X . It will always be clear from the context whether we consider X as an abstract variety or as a complex manifold. Burgos introduces in [Bur94] a differential graded algebra $A_{\overline{X}}(X, \log D)$ of complex valued smooth differential forms on X with logarithmic singularities along D . It has a real subcomplex

$$A_{\overline{X}, \mathbb{R}}(X, \log D) \subset A_{\overline{X}}(X, \log D) ,$$

whose complexification is $A_{\bar{X}}(X, \log D)$, with an increasing weight filtration

$$\mathcal{W}_* A_{\bar{X}, \mathbb{R}}(X, \log D) \quad (12)$$

inducing a weight filtration on $A_{\bar{X}}(X, \log D)$ by complexification. Furthermore, the complex $A_{\bar{X}}(X, \log D)$ has a decreasing Hodge filtration

$$\mathcal{F}^* A_{\bar{X}}(X, \log D) . \quad (13)$$

Let

$$\iota : A_{\bar{X}, \mathbb{R}}(X, \log D) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} A_{\bar{X}}(X, \log D)$$

be the canonical identification. The triple

$$((A_{\bar{X}, \mathbb{R}}(X, \log D), \mathcal{W}_*), (A_{\bar{X}}(X, \log D), \mathcal{W}_*, \mathcal{F}^*), \iota) \quad (14)$$

is a mixed \mathbb{R} -Hodge complex in the sense of [Del74, 8.1.5] ([Bur94, Corollary 2.2]). Recall that this implies in particular that the cohomology $H^*(A_{\bar{X}, \mathbb{R}}(X, \log D))$ carries a mixed \mathbb{R} -Hodge structure as recalled below.

A construction of these objects will be explained in a more general situation in Subsection 4.2 below. In order to get the complexes above from this general case one must specialize the manifold part to be a point.

We will use the fact that the embeddings

$$A_{\bar{X}}(X, \log D) \rightarrow A(X) \quad \text{and} \quad A_{\bar{X}, \mathbb{R}}(X, \log D) \rightarrow A_{\mathbb{R}}(X) \quad (15)$$

are quasi-isomorphisms by [Bur94, Theorem 2.1] together with [Del71, 3.1.8]. So the cohomology $H^n(X; \mathbb{R})$ carries a mixed \mathbb{R} -Hodge structure as follows [Del71, 3.2.5]: The weight filtration is induced via (15) by the filtration (12) (shifted by n):

$$\mathcal{W}_k H^n(X; \mathbb{R}) := \text{im} \left(H^n(\mathcal{W}_{k-n} A_{\bar{X}, \mathbb{R}}(X, \log D)) \rightarrow H^n(A_{\bar{X}, \mathbb{R}}(X, \log D)) \right) ,$$

and the Hodge filtration on the complexification $H^n(X; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^n(X; \mathbb{C})$ is induced via (15) by (13).

Recall that the décalage of the weight filtration (12) is given by

$$\hat{\mathcal{W}}_k A_{\bar{X}, \mathbb{R}}^n(X, \log D) := \{ \omega \in \mathcal{W}_{k-n} A_{\bar{X}, \mathbb{R}}^n(X, \log D) \mid d\omega \in \mathcal{W}_{k-n-1} A_{\bar{X}, \mathbb{R}}^{n+1}(X, \log D) \} . \quad (16)$$

It also induces the weight filtration on $H^*(X, \mathbb{R})$ (see [Del71, 1.3.4]), and by [Del71, 3.2.10] the associated spectral sequence degenerates at E_1 . In particular, we have

$$\mathcal{W}_k H^*(X, \mathbb{R}) \cong H^*(\hat{\mathcal{W}}_k A_{\bar{X}, \mathbb{R}}(X, \log D)). \quad (17)$$

The same reasoning applies to the cohomology of any mixed \mathbb{R} -Hodge complex (see [Del74, 8.1.9]).

In the following we get rid of the choice of the good compactification and define a presheaf of mixed \mathbb{R} -Hodge complexes on the site $\mathbf{Sm}_{\mathbb{C}}$. For a smooth variety X the category I_X

of good compactifications of X with respect to maps under X is cofiltered and essentially small. For a morphism

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ \overline{X} & \longrightarrow & \overline{X}' \end{array}$$

in I_X we have an inclusion

$$A_{\overline{X}'}(X, \log(\overline{X}' \setminus X)) \subseteq A_{\overline{X}}(X, \log(\overline{X} \setminus X)) \quad (18)$$

which is compatible with the real subcomplex and the weight and Hodge filtrations. By [Del71, Thm. 3.2.5] and (17) it is in fact a bifiltered quasi-isomorphism with respect to $\hat{\mathcal{W}}$ and \mathcal{F} . Hence we get functors

$$I_X^{op} \rightarrow \mathbf{Ch}, \quad (X \rightarrow \overline{X}) \mapsto A_{\overline{X}}(X, \log(\overline{X} \setminus X)) \quad (19)$$

and subfunctors corresponding to the real subcomplex and the weight and Hodge filtrations.

Definition 3.1. *We define the presheaf of chain complexes*

$$A_{\log} \in \mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Sm}_{\mathbb{C}})$$

by

$$X \mapsto A_{\log}(X) := \operatorname{colim}_{I_X^{op}} A_{\overline{X}}(X, \log(\overline{X} \setminus X)) .$$

Furthermore, we define

$$A_{\log, \mathbb{R}}, \hat{\mathcal{W}}_* A_{\log, \mathbb{R}}, \hat{\mathcal{W}}_* A_{\log}, \mathcal{F}^* A_{\log} \in \mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Sm}_{\mathbb{C}})$$

in a similar manner.

Using the exactness of a filtered colimit it is clear that for any $X \in \mathbf{Sm}_{\mathbb{C}}$ the triple

$$((A_{\log, \mathbb{R}}(X), \mathcal{W}_*), (A_{\log}(X), \mathcal{W}_*, \mathcal{F}^*), \iota)$$

is a mixed \mathbb{R} -Hodge complex computing the mixed \mathbb{R} -Hodge structure on $H^*(X; \mathbb{R})$ described above which moreover depends in a functorial way on X .

The presheaves of Definition 3.1 can be seen as objects of the ∞ -category $\mathbf{Fun}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(\mathbf{Ch}))$ or also as objects of $\mathbf{Fun}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(\mathbf{Ch})[W^{-1}])$. We claim that they are in fact sheaves when considered as objects in the latter, i.e. that they satisfy Zariski descent (see Definition A.1).

Lemma 3.2. *We have*

$$A_{\log}, A_{\log, \mathbb{R}}, \hat{\mathcal{W}}_k A_{\log}, \mathcal{F}^p \cap \hat{\mathcal{W}}_k A_{\log} \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(\mathbf{Ch})[W^{-1}]) .$$

Proof. The functors A and $A_{\mathbb{R}}$ satisfy descent in the analytic, and hence in the Zariski topology. Because of the quasi-isomorphisms (15) the inclusions

$$A_{\log}(X) \rightarrow A(X) , \quad A_{\log, \mathbb{R}}(X) \rightarrow A_{\mathbb{R}}(X) \quad (20)$$

are quasi-isomorphisms for all $X \in \mathbf{Sm}_{\mathbb{C}}$. Hence, the subfunctors A_{\log} and $A_{\log, \mathbb{R}}$ satisfy Zariski descent, too.

Next we handle $\hat{\mathcal{W}}_k$. We let

$$\mathbf{tot} : \mathbf{cCh} \rightarrow \mathbf{Ch} \quad (21)$$

be the total complex functor from cosimplicial chain complexes to chain complexes. Let U_{\bullet} be the Čech nerve of a Zariski cover $U \rightarrow X$. We have to show that the map

$$\hat{\mathcal{W}}_k A_{\log, \mathbb{R}}(X) \rightarrow \mathbf{tot} \left(\hat{\mathcal{W}}_k A_{\log, \mathbb{R}}(U_{\bullet}) \right) \quad (22)$$

is a quasi-isomorphism. Now

$$((A_{\log, \mathbb{R}}(U_{\bullet}), \mathcal{W}_*), (A_{\log}(U_{\bullet}), \mathcal{W}_*, \mathcal{F}^*), \iota)$$

is a cosimplicial mixed \mathbb{R} -Hodge complex. Deligne has shown [Del74, 8.1.15] that the associated total complex $\mathbf{tot} A_{\log, \mathbb{R}}(U_{\bullet})$ inherits the structure of a mixed \mathbb{R} -Hodge complex whose weight and Hodge filtrations \mathcal{W}_* and \mathcal{F}^* are given by

$$\mathcal{W}_k(\mathbf{tot} A_{\log, \mathbb{R}}(U_{\bullet})) := \bigoplus_p \mathcal{W}_{k+p} A_{\log, \mathbb{R}}(U_p) \quad \text{and} \quad \mathcal{F}^p(\mathbf{tot} A_{\log}(U_{\bullet})) := \mathbf{tot}(\mathcal{F}^p A_{\log}(U_{\bullet})),$$

respectively. Hence the cohomology $H^*(\mathbf{tot} A_{\log, \mathbb{R}}(U_{\bullet}))$ carries a mixed \mathbb{R} -Hodge structure and by (17) and the remark following it the horizontal maps in the diagram

$$\begin{array}{ccc} H^* \left(\hat{\mathcal{W}}_k (\mathbf{tot} A_{\log, \mathbb{R}}(U_{\bullet})) \right) & \xrightarrow{\cong} & \mathcal{W}_k H^* (\mathbf{tot} A_{\log, \mathbb{R}}(U_{\bullet})) \\ \uparrow & & \uparrow \\ H^* \left(\hat{\mathcal{W}}_k A_{\log, \mathbb{R}}(X) \right) & \xrightarrow{\cong} & \mathcal{W}_k H^* (A_{\log, \mathbb{R}}(X)) \end{array} \quad (23)$$

are isomorphisms.

It is easy to see that the décalage of the weight filtration satisfies

$$\hat{\mathcal{W}}_k (\mathbf{tot} A_{\log, \mathbb{R}}(U_{\bullet})) = \mathbf{tot} \left(\hat{\mathcal{W}}_k A_{\log, \mathbb{R}}(U_{\bullet}) \right) .$$

Hence, in order to prove that (22) is a quasi-isomorphism it suffices to check that the right vertical map in (23) is an isomorphism. But $H^*(A_{\log, \mathbb{R}}(X)) \rightarrow H^*(\mathbf{tot} A_{\log, \mathbb{R}}(U_{\bullet}))$ is a morphism of mixed \mathbb{R} -Hodge structures, hence strict with respect to both filtrations [Del71, 1.2.10], and an isomorphism by Zariski descent for $A_{\log, \mathbb{R}}$.

The main non-formal input for this argument was (17), which is a consequence of the fact that the spectral sequence associated to $\hat{\mathcal{W}}_*$ degenerates at E_1 . Since this holds true for $\mathcal{F}^* A_{\log}$ and the induced filtration $\hat{\mathcal{W}}_k \cap \mathcal{F}^* A_{\log}$ we can argue similarly for the remaining cases. \square

Definition 3.3. For $p \in \mathbb{Z}$ we define the absolute Hodge complex twisted by p as

$$\mathbf{DR}_{\mathbb{C}}(p) := \text{Cone} \left((2\pi i)^p \hat{\mathcal{W}}_{2p} A_{\log, \mathbb{R}} \oplus \hat{\mathcal{W}}_{2p} \cap \mathcal{F}^p A_{\log} \rightarrow \hat{\mathcal{W}}_{2p} A_{\log} \right) [2p - 1] \in \mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Sm}_{\mathbb{C}})$$

where the map defining the cone is given by $(\omega, \eta) \mapsto \omega - \eta$. Furthermore, we set

$$\mathbf{DR}_{\mathbb{C}} := \prod_{p \geq 0} \mathbf{DR}_{\mathbb{C}}(p) .$$

We refer to A.11 for the conventions concerning the cone. Note that by Lemma 3.2 we can consider

$$\mathbf{DR}_{\mathbb{C}}, \mathbf{DR}_{\mathbb{C}}(p) \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(\mathbf{Ch})[W^{-1}]) . \quad (24)$$

The cohomology of $\mathbf{DR}(p)(X)$ is, up to a shift, the absolute Hodge cohomology of X as defined by Beilinson in [Beĭ86]:

$$H^k(\mathbf{DR}_{\mathbb{C}}(p)(X)) \cong H_{\text{Hodge}}^{k+2p}(X, \mathbb{R}(p)) . \quad (25)$$

3.2 Arithmetic sites and the absolute Hodge complex $\mathbf{DR}_{\mathbb{Z}}$

In Subsection 3.1 we described the absolute Hodge complex $\mathbf{DR}_{\mathbb{C}}(X)$ of a smooth complex algebraic variety X . In the present subsection we extend this construction to regular and separated schemes of finite type X over $\mathbf{Spec}(\mathbb{Z})$. The basic idea is to apply $\mathbf{DR}_{\mathbb{C}}$ to the complexification $X \otimes \mathbb{C} := \mathbf{Spec}(\mathbb{C}) \times_{\mathbf{Spec}(\mathbb{Z})} X$, which is a smooth variety over \mathbb{C} , and to take the action of complex conjugation into account.

We let $\mathbf{Reg}_{\mathbb{Z}}$ be the site of regular separated schemes of finite type over $\mathbf{Spec}(\mathbb{Z})$ with the topology given by Zariski open coverings. We further introduce the ∞ -category

$$\mathbf{S}_{\mathbb{Z}} := \mathbf{N}(\mathbf{Reg}_{\mathbb{Z}}^{op}) .$$

The group $\mathbf{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ acts by complex conjugation on the site $\mathbf{Sm}_{\mathbb{C}}$ so that the complex conjugate of the complex manifold X is the complex manifold \bar{X} . Here \bar{X} has the same underlying smooth manifold as X but the opposite complex structure. If X is complex algebraic, then so is \bar{X} .

The base-change functor

$$B : \mathbf{Reg}_{\mathbb{Z}} \rightarrow \mathbf{Sm}_{\mathbb{C}} , \quad B(X) := X \otimes \mathbb{C}$$

is compatible with the topologies and $\mathbb{Z}/2\mathbb{Z}$ -invariant in the sense that there exists a natural isomorphism

$$u_X : B(X) \xrightarrow{\sim} \overline{B(X)} \quad (26)$$

given again by complex conjugation.

An equivariant structure on a presheaf $F \in \mathbf{PSh}(\mathbf{Sm}_{\mathbb{C}})$ is given by a natural isomorphism $c_X : F(X) \rightarrow F(\bar{X})$ for all $X \in \mathbf{Sm}_{\mathbb{C}}$ such that $c_{\bar{X}} \circ c_X = \text{id}$. For example, the sheaf of complex differential forms A is equivariant by $c_X : A(X) \xrightarrow{\sim} A(\bar{X})$, $c_X(\omega) := \bar{\omega}$. This

action clearly preserves the real subspace $A_{\mathbb{R}}(X)$, restricts to $c_X : A_{\log, \mathbb{R}}(X) \xrightarrow{\sim} A_{\log, \mathbb{R}}(\bar{X})$ and preserves the weight and Hodge filtrations. Hence it induces an equivariant structure on the absolute Hodge complexes $\mathbf{DR}_{\mathbb{C}}(p)$ and their product $\mathbf{DR}_{\mathbb{C}}$.

If a presheaf $F \in \mathbf{PSh}(\mathbf{Sm}_{\mathbb{C}})$ is equivariant, then the pull-back $B^*F \in \mathbf{PSh}(\mathbf{Reg}_{\mathbb{Z}})$ has a natural $\mathbb{Z}/2\mathbb{Z}$ -action given by $\omega \mapsto u_X^* c_{B(X)}(\omega)$ for $\omega \in F(B(X))$.

If U is a set with an action of a group G , then we let U^G denote the subset of fixed points. In the case of presheaves of sets with G -action we take the invariants objectwise.

Definition 3.4. *We define the arithmetic versions of the absolute Hodge complex by*

$$\mathbf{DR}_{\mathbb{Z}}(p) := (B^* \mathbf{DR}_{\mathbb{C}}(p))^{\mathbb{Z}/2\mathbb{Z}} \in \mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Reg}_{\mathbb{Z}})$$

and

$$\mathbf{DR}_{\mathbb{Z}} := (B^* \mathbf{DR}_{\mathbb{C}})^{\mathbb{Z}/2\mathbb{Z}} \in \mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Reg}_{\mathbb{Z}}) .$$

Note that $\mathbf{DR}_{\mathbb{Z}} \cong \prod_{p \geq 0} \mathbf{DR}_{\mathbb{Z}}(p)$. Furthermore, since these presheaves are in fact presheaves of complexes of \mathbb{R} -vector spaces the operation of taking invariants under $\mathbb{Z}/2\mathbb{Z}$ preserves descent. It follows from (24) that the arithmetic versions of the absolute Hodge complexes have the descent property, too:

$$\mathbf{DR}_{\mathbb{Z}}(p), \mathbf{DR}_{\mathbb{Z}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{Z}}, \mathbf{N}(\mathbf{Ch})[W^{-1}]) .$$

3.3 K -theory

In this subsection we discuss the definition of algebraic K -theory in the complex and the arithmetic case.

Definition 3.5. *We define the sheaves*

$$\mathbf{K}_{\mathbb{C}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(\mathbf{Sp})[W^{-1}]) , \quad \mathbf{K}_{\mathbb{Z}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{Z}}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$$

by specializing in the definitions of $\mathbf{K}_{Mf, \mathbb{C}}$ and $\mathbf{K}_{Mf, \mathbb{Z}}$ given in 4.15 along the canonical maps $\mathbf{S}_{\mathbb{C}} \rightarrow \mathbf{S}_{Mf, \mathbb{C}}$ and $\mathbf{S}_{\mathbb{Z}} \rightarrow \mathbf{S}_{Mf, \mathbb{Z}}$, respectively.

In the following we explain why this definition reproduces the standard version of algebraic K -theory, i.e., Quillen's K [Qui73] or Thomason's K^B [TT90, 6.4]. In order to be specific we consider the arithmetic case. We can write

$$\mathbf{K}_{\mathbb{Z}} \cong L(K(i\mathbf{Vect}_{\mathbb{Z}})) , \tag{27}$$

where

$$i\mathbf{Vect}_{\mathbb{Z}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{Z}}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]))$$

is obtained from $i\mathbf{Vect}_{Mf, \mathbb{Z}}$ again by specializing along $\mathbf{S}_{\mathbb{Z}} \rightarrow \mathbf{S}_{Mf, \mathbb{Z}}$, K is the K -theory functor from 2.1 and L is the sheafification functor (122).

We consider the subcategory of regular affine schemes $\mathbf{Reg}_{\mathbb{Z}}^{aff} \subset \mathbf{Reg}_{\mathbb{Z}}$ and the corresponding inclusion of nerves $\mathbf{S}_{\mathbb{Z}}^{aff} \subset \mathbf{S}_{\mathbb{Z}}$. If $\mathbf{Spec}(R) \in \mathbf{Reg}_{\mathbb{Z}}^{aff}$, then $i\mathbf{Vect}_{\mathbb{Z}}(\mathbf{Spec}(R))$ is equivalent to the symmetric monoidal groupoid of finitely generated projective R -modules

$i\mathcal{P}(R)$ (with respect to the direct sum). It is well-known, that the usual K -theory spectrum of the ring R can be defined in terms of group completion as $K(\mathcal{P}(R))$ (see e.g. [Wei, IV.4.8, IV.4.11.1]). It follows that the functor

$$K(i\mathbf{Vect}_{\mathbb{Z}})_{|\mathbf{S}_{\mathbb{Z}}^{aff}} \in \mathbf{Fun}(\mathbf{S}_{\mathbb{Z}}^{aff}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$$

associates to each object $\mathbf{Spec}(R) \in \mathbf{S}_{\mathbb{Z}}^{aff}$ the usual K -theory spectrum. It is further known that $K(i\mathbf{Vect}_{\mathbb{Z}})_{|\mathbf{S}_{\mathbb{Z}}^{aff}}$ satisfies Zariski descent (see below). The latter property allows to drop the sheafification in the definition (27) of $\mathbf{K}_{\mathbb{Z}}$ restricted to $\mathbf{S}_{\mathbb{Z}}^{aff}$ and we have an equivalence

$$(\mathbf{K}_{\mathbb{Z}})_{|\mathbf{S}_{\mathbb{Z}}^{aff}} \cong K(i\mathbf{Vect}_{\mathbb{Z}})_{|\mathbf{S}_{\mathbb{Z}}^{aff}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{Z}}^{aff}, \mathbf{N}(\mathbf{Sp})[W^{-1}]) .$$

If $X \in \mathbf{Reg}_{\mathbb{Z}}$ is not affine, an exact sequence of vector bundles on X does not split in general. Because of this the K -theory of X is usually not defined by group completion, but by applying a different K -theory machine to the exact category of vector bundles. This construction leads to a functor

$$\mathbf{K}'_{\mathbb{Z}} \in \mathbf{Fun}(\mathbf{S}_{\mathbb{Z}}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$$

which we consider as the standard definition of algebraic K -theory. Since the schemes in $\mathbf{Reg}_{\mathbb{Z}}$ are separated, noetherian, and regular, the K -theory of vector bundles coincides with Thomason's K^B -theory which satisfies Zariski descent [TT90, 7.6, 8.4]. We conclude that

$$\mathbf{K}'_{\mathbb{Z}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{Z}}, \mathbf{N}(\mathbf{Sp})[W^{-1}]) .$$

Since every scheme $X \in \mathbf{Reg}_{\mathbb{Z}}$ admits a Zariski covering by affines we conclude that the restriction

$$\mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{Z}}, \mathbf{C}) \rightarrow \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{Z}}^{aff}, \mathbf{C})$$

is an equivalence for every presentable target category \mathbf{C} . In particular, since

$$(\mathbf{K}_{\mathbb{Z}})_{|\mathbf{S}_{\mathbb{Z}}^{aff}} \cong (\mathbf{K}'_{\mathbb{Z}})_{|\mathbf{S}_{\mathbb{Z}}^{aff}} ,$$

we conclude that $\mathbf{K}_{\mathbb{Z}} \cong \mathbf{K}'_{\mathbb{Z}}$.

3.4 Regulators

We start with the definition of the regulators in the complex and the arithmetic cases.

Definition 3.6. *We define the complex and arithmetic versions of the Beilinson regulator*

$$\mathbf{r}_{\mathbb{C}}^{\text{Beil}} : \mathbf{K}_{\mathbb{C}} \rightarrow H(\mathbf{DR}_{\mathbb{C}}) , \quad \mathbf{r}_{\mathbb{Z}}^{\text{Beil}} : \mathbf{K}_{\mathbb{Z}} \rightarrow H(\mathbf{DR}_{\mathbb{Z}})$$

by specializing the regulators defined in 4.20 and 4.22 along $\mathbf{S}_{\mathbb{C}} \rightarrow \mathbf{S}_{Mf, \mathbb{C}}$ and $\mathbf{S}_{\mathbb{Z}} \rightarrow \mathbf{S}_{Mf, \mathbb{Z}}$, respectively.

The main goal of the the present subsection is to give an argument why the regulators introduced in Definition 3.6 coincide with the usual ones, introduced originally by Beilinson [Bei84, Bei86]. This justifies in particular the formulation of Beilinson's conjectures in the following subsection. The remaining part of this subsection may be skipped on first reading.

We consider the complex case. We let $i\mathbf{Vect}_{\mathbb{C}}$ be the symmetric monoidal stack on $\mathbf{Sm}_{\mathbb{C}}$ of bundles obtained from $i\mathbf{Vect}_{Mf, \mathbb{C}}$ introduced in Definition 4.14 by restriction along $\mathbf{S}_{\mathbb{C}} \rightarrow \mathbf{S}_{Mf, \mathbb{C}}$. We abbreviate its nerve by

$$M := \mathbf{N}(i\mathbf{Vect}_{\mathbb{C}}) \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{C}}, \mathbf{CommMon}(\mathbf{N}(\mathbf{sSet})[W^{-1}])) .$$

We further consider the sheaf of chain complexes

$$\mathbf{DR}_{\mathbb{C}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(\mathbf{Ch})[W^{-1}])$$

(see Definition 3.3).

Definition 3.7. *We define the primitive part of the cohomology of M with coefficients in the sheaf of chain complexes $\mathbf{DR}_{\mathbb{C}}(p)$ by*

$$\mathbf{Prim}(M, \mathbf{DR}_{\mathbb{C}}(p)) := \pi_0(\underline{\mathbf{Map}}(M, \Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p))) ,$$

where $\underline{\mathbf{Map}}(X, Y) \in \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}])$ denotes the structured mapping object for $X \in \mathbf{CommMon}(\mathbf{N}(\mathbf{sSet})[W^{-1}])$ and $Y \in \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}])$.

The algebraic K -theory sheaf $\mathbf{K}_{\mathbb{C}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$ given in Definitions 3.5 and 4.15 can be presented as

$$\mathbf{K}_{\mathbb{C}} \cong L(\mathbf{sp}(\Omega B(M))) .$$

We have equivalences

$$\Omega^{\infty} \mathbf{Map}(\mathbf{K}_{\mathbb{C}}, H(\mathbf{DR}_{\mathbb{C}}(p))) \cong \underline{\mathbf{Map}}(\Omega B(M), \Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p))) \cong \underline{\mathbf{Map}}(M, \Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p))) . \quad (28)$$

In view of (28) the p -component of the regulator $\mathbf{r}_{\mathbb{C}}^{\text{Beil}} \in \pi_0(\Omega^{\infty} \mathbf{Map}(\mathbf{K}_{\mathbb{C}}, H(\mathbf{DR}_{\mathbb{C}})))$ is characterized uniquely by the corresponding primitive cohomology class which we will denote by $c_{\omega(p)} \in \mathbf{Prim}(M, \mathbf{DR}_{\mathbb{C}}(p))$.

The goal of comparison of $\mathbf{r}_{\mathbb{C}}^{\text{Beil}}$ with the usual definitions is achieved by a calculation of $\mathbf{Prim}(M, \mathbf{DR}_{\mathbb{C}}(p))$ in terms of the absolute Hodge cohomology of the simplicial varieties $BGL_{\bullet}(n)$ (see below) and the identification of $c_{\omega(p)}$ with the $2p$ -component of the Chern character class. Due to Deligne's computations, the absolute Hodge cohomology of $BGL_{\bullet}(n)$ is well understood.

The stack $i\mathbf{Vect}_{\mathbb{C}}$ has an atlas

$$\mathbb{N}_0 \rightarrow i\mathbf{Vect}_{\mathbb{C}}$$

which represents the vector bundle $\bigsqcup_{n \in \mathbb{N}_0} \mathbb{C}^n \rightarrow \mathbb{N}_0$. The atlas gives rise to a groupoid

$$GL := \left(\bigsqcup_{n \in \mathbb{N}_0} GL(n) \rightrightarrows \mathbb{N}_0 \right)$$

in $\mathbf{Sm}_{\mathbb{C}}$. We let $BGL_{\bullet} := \mathbb{N}(GL) \in (\mathbf{Sm}_{\mathbb{C}})^{\Delta^{op}}$ be its nerve. It is the disjoint union of simplicial varieties $\coprod_{n \in \mathbb{N}_0} BGL_{\bullet}(n)$. In particular, we can consider its absolute Hodge cohomology

$$H^0(\mathbf{DR}_{\mathbb{C}}(p)(BGL_{\bullet})) \cong H_{\text{Hodge}}^{2p}(BGL_{\bullet}, \mathbb{R}(p)) \cong \prod_{n \in \mathbb{N}_0} H_{\text{Hodge}}^{2p}(BGL_{\bullet}(n), \mathbb{R}(p)).$$

Proposition 3.8. *The primitive cohomology $\text{Prim}(M, \mathbf{DR}_{\mathbb{C}}(p))$ is naturally isomorphic to the one-dimensional real subspace*

$$\mathbb{R} \cdot (s_{p,0}, s_{p,1}, s_{p,2}, \dots) \subseteq \prod_{n \in \mathbb{N}_0} H_{\text{Hodge}}^{2p}(BGL_{\bullet}(n), \mathbb{R}(p))$$

spanned by the degree $2p$ component of the universal Chern character class in absolute Hodge cohomology (see the proof for notation). Under this isomorphism $c_{\omega(p)}$ corresponds to the universal Chern character class.

This implies in particular that on the level of homotopy groups $\mathbf{r}_{\mathbb{C}}^{\text{Beil}}$ and similarly $\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}$ induce Beilinson's regulator introduced originally in [Beil84, 2.3].

Proof. Let

$$Y : (\mathbf{Sm}_{\mathbb{C}})^{\Delta^{op}} \rightarrow \mathbf{Sh}_{\mathbf{sSet}}(\mathbf{Sm}_{\mathbb{C}})$$

be the Yoneda embedding, $\kappa : \mathbf{Sh}_{\mathbf{sSet}}(\mathbf{Sm}_{\mathbb{C}}) \rightarrow \mathbf{Fun}(\mathbf{S}_{\mathbb{C}}, \mathbb{N}(\mathbf{sSet})[W^{-1}])$ the canonical map, and

$$L : \mathbf{Fun}(\mathbf{S}_{\mathbb{C}}, \mathbb{N}(\mathbf{sSet})[W^{-1}]) \rightarrow \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{C}}, \mathbb{N}(\mathbf{sSet})[W^{-1}])$$

the sheafification functor (122). Then we have an equivalence

$$L(\kappa(Y(BGL_{\bullet}))) \cong M$$

in $\mathbf{Fun}^{desc}(\mathbf{S}, \mathbb{N}(\mathbf{sSet})[W^{-1}])$. Since $\Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p))$ satisfies descent this induces an equivalence of mapping spaces between objects of $\mathbf{Fun}(\mathbf{S}_{\mathbb{C}}, \mathbb{N}(\mathbf{sSet})[W^{-1}])$

$$\text{map}(M, \Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p))) \cong \text{map}(\kappa(Y(BGL_{\bullet})), \Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p))). \quad (29)$$

In the appendix (Subsection A.6) we prove the technical fact that for every $k \geq 0$ we have a natural isomorphism

$$\pi_k(\text{map}(\kappa(Y(BGL_{\bullet})), \Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p)))) \cong H^{-k}(\text{tot} \mathbf{DR}_{\mathbb{C}}(p)(BGL_{\bullet})). \quad (30)$$

By construction and (25) the right hand side is the absolute Hodge cohomology of the simplicial variety BGL_{\bullet} :

$$H^{-k}(\text{tot} \mathbf{DR}_{\mathbb{C}}(p)(BGL_{\bullet})) \cong \prod_{n \in \mathbb{N}} H_{\text{Hodge}}^{2p-k}(BGL_{\bullet}(n), \mathbb{R}(p)). \quad (31)$$

According to Deligne [Del74, 9.1.1] we have $H^{2k+1}(BGL_{\bullet}(n), \mathbb{R}) = 0$ and the real Hodge structure on $H^{2k}(BGL_{\bullet}(n), \mathbb{R})$ is pure of type (k, k) , i.e. the weight filtration has a single step at $2k$ and the Hodge filtration a single step at k . Using this and the definition of $\mathbf{DR}_{\mathbb{C}}(p)$ as a cone one sees that

$$H_{\text{Hodge}}^k(BGL_{\bullet}(n), \mathbb{R}(p)) \cong H^k(BGL_{\bullet}(n), \mathbb{R}(p)) , \quad 0 \leq p \leq \frac{k}{2}, k \text{ even}, \quad (32)$$

and all other absolute Hodge cohomology groups of $BGL_{\bullet}(n)$ vanish. In particular, combining (29), (30), and (31) we get

$$\pi_0(\text{map}(M, \Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p)))) \cong \prod_{n \in \mathbb{N}} H^{2p}(BGL_{\bullet}(n), \mathbb{R}(p)). \quad (33)$$

It is known that the cohomology of $BGL_{\bullet}(n)$ is a polynomial ring

$$H^{2*}(BGL_{\bullet}(n), \mathbb{R}(*)) = \mathbb{R}[s_{1,n}, \dots, s_{n,n}]$$

where $\frac{1}{p!} s_{p,n} \in H^{2p}(BGL_{\bullet}(n), \mathbb{R}(p))$ is the component of the Chern character in degree $2p$. Moreover, for the map $\mu : BGL_{\bullet}(n) \times BGL_{\bullet}(m) \rightarrow BGL_{\bullet}(n+m)$ induced by the direct sum, we have

$$\mu^*(s_{k,n+m}) = \text{pr}_1^* s_{k,n} + \text{pr}_2^* s_{k,m}.$$

This, together with the fact that the space of primitive elements intersects trivially with the space of decomposable elements and an easy computation imply, that the subgroup of primitives $H^0(\text{tot} \mathbf{DR}_{\mathbb{C}}(p)(BGL_{\bullet}))^{prim}$, defined in (120), is the one-dimensional \mathbb{R} -vector space spanned by the class

$$s_p := (s_{p,1}, s_{p,2}, \dots) ,$$

where we denote the preimage of $s_{p,n}$ in $H^0(\text{tot} \mathbf{DR}_{\mathbb{C}}(p)(BGL_{\bullet}(n)))$ under the isomorphisms (31) and (32) by the same symbol.

We have an injection (cf. (121))

$$\text{Prim}(M, \mathbf{DR}_{\mathbb{C}}(p)) \hookrightarrow H^0(\text{tot} \mathbf{DR}_{\mathbb{C}}(p)(BGL_{\bullet}))^{prim} . \quad (34)$$

It follows from the construction of $\omega(p)$ in Subsection 4.4 below that the element $c_{\omega(p)} \in \text{Prim}(M, \mathbf{DR}_{\mathbb{C}}(p))$ corresponding to the p -component of $\mathbf{r}_{\mathbb{C}}^{\text{Beil}}$ goes to the class $\frac{1}{p!} s_p$ under the map (34). This shows surjectivity of (34) and finishes the proof of the proposition. \square

3.5 Beilinson's conjectures

In this Subsection we describe the conjectural picture of kernel and cokernel of the regulator map $\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}$ as given by the conjectures of Beilinson and Parshin [Beĭ84, Beĭ86, §8]. We fix a scheme $X \in \mathbf{Reg}_{\mathbb{Z}}$, proper and flat over $\text{Spec}(\mathbb{Z})$, which has potentially good reduction at every prime. For example, X could be smooth and proper over the ring of integers in some number field.

Then we have homomorphisms

$$\mathbf{r}_{\mathbb{Z}}^{\text{Beil}} : \pi_i(\mathbf{K}_{\mathbb{Z}}(X)) \rightarrow H^{-i}(\mathbf{DR}_{\mathbb{Z}}(X))$$

for all $i \geq 0$.

Conjecture 3.9. 1. For $i \geq 2$ the regulator induces an isomorphism

$$\pi_i(\mathbf{K}_{\mathbb{Z}}(X)) \otimes \mathbb{R} \xrightarrow{\cong} H^{-i}(\mathbf{DR}_{\mathbb{Z}}(X)) . \quad (35)$$

2. For $i = 1$ (35) is expected to be injective. Let $N^p(X_{\mathbb{Q}})$ be the group of codimension p -cycles in the generic fibre $X_{\mathbb{Q}}$ modulo homological equivalence and $N(X_{\mathbb{Q}}) := \prod_{p \geq 0} N^p(X_{\mathbb{Q}})$. Then the cycle map in de Rham cohomology

$$N^p(X_{\mathbb{Q}}) \hookrightarrow H^{2p}(B(X); \Omega_{B(X)}^{\bullet}) \cong H^{2p}(\hat{\mathcal{W}}_{2p} A_{\log}(B(X)))$$

together with the natural map $H^{2p}(\hat{\mathcal{W}}_{2p} A_{\log}(B(X))) \rightarrow H^{-1}(\mathbf{DR}_{\mathbb{Z}}(p+1)(X))$ coming from the definition of $\mathbf{DR}_{\mathbb{Z}}(p)$ as a cone induce an injection

$$\bar{z}^p : N^p(X_{\mathbb{Q}}) \hookrightarrow H^{-1}(\mathbf{DR}_{\mathbb{Z}}(p+1)(X)) .$$

We let $\bar{z} := \prod_{p \geq 0} \bar{z}^p$. It is expected that

$$\mathbf{r}_{\mathbb{Z}}^{\text{Beil}} \oplus \bar{z} : \pi_1(\mathbf{K}_{\mathbb{Z}}(X)) \otimes \mathbb{R} \oplus N(X_{\mathbb{Q}}) \otimes \mathbb{R} \rightarrow H^{-1}(\mathbf{DR}_{\mathbb{Z}}(X))$$

is an isomorphism.

3. For $i = 0$ we have

$$H^0(\mathbf{DR}_{\mathbb{Z}}(p)(X)) \cong (H^{2p}(X \otimes_{\mathbb{Z}} \mathbb{C}; \mathbb{R}(p)) \cap \mathcal{F}^p H^{2p}(X \otimes_{\mathbb{Z}} \mathbb{C}; \mathbb{C}))^{\mathbb{Z}/2\mathbb{Z}} .$$

Hence $H^0(\mathbf{DR}_{\mathbb{Z}}(X))$ is the group of real Hodge classes on $X \otimes_{\mathbb{Z}} \mathbb{C}$. As the example of number rings shows the map $\pi_0(\mathbf{K}_{\mathbb{Z}}(X)) \otimes \mathbb{R} \rightarrow H^0(\mathbf{DR}_{\mathbb{Z}}(X))$ can be far from being surjective.

At the moment, these conjectures are far out of reach. Essentially, the only known case is that of the ring of integers in a number field where they are due to Borel. Also, one can weaken the assumptions on X but then the statement of the conjectures becomes more complicated.

3.6 The relative Chern character

An alternative approach to regulators goes back to Karoubi [Kar83, Kar87] using his relative Chern character (see below). This was further studied and generalized in [Tam12]. Roughly, the idea is as follows: We consider a complex algebraic variety X . Then we have its algebraic K -theory $\mathbf{K}_{\mathbb{C}}(X)$, the topological K -theory of its associated complex

manifold $\mathbf{K}^{top}(X)$, and a natural comparison map $c : \mathbf{K}_{\mathbb{C}}(X) \rightarrow \mathbf{K}^{top}(X)$. If we compose the usual Chern character \mathbf{ch} from topological K -theory to singular or de Rham cohomology with c , the resulting map on algebraic K -theory contains only trivial information. Instead, we define the relative K -theory $\mathbf{K}^{rel}(X)$ as the homotopy fibre of the map c . The relative Chern character \mathbf{ch}^{rel} is defined on the relative K -theory and maps $\pi_i(\mathbf{K}^{rel}(X))$ to $\prod_p H^{2p-i-1}(X; \mathbb{C})/\mathcal{F}^p$, the de Rham cohomology divided by the Hodge filtration. This map now carries interesting secondary information.

We have a natural map from $H^{2p-i-1}(X; \mathbb{C})/\mathcal{F}^p$ to weak absolute Hodge cohomology $H^{2p-i}(\mathbf{D}(p)(X))$, where the complex $\mathbf{D}(p)$ is defined similar as the absolute Hodge complex $\mathbf{DR}_{\mathbb{C}}(p)$ but discarding the weight filtration (see (38)). The analog of Beilinson's regulator for complex algebraic varieties may be viewed as a map $\mathbf{r}_{\mathbb{C}, \mathbf{D}, p}^{\text{Beil}} : \pi_i(\mathbf{K}_{\mathbb{C}}(X)) \rightarrow H^{2p-i}(\mathbf{D}(p)(X))$ (see (39)).

Theorem 3.10 ([Tam12, Thm. 3.9]). *For all $p \geq 0$ the diagram*

$$\begin{array}{ccc} \pi_i(\mathbf{K}^{rel}(X)) & \xrightarrow{\mathbf{ch}_p^{rel}} & H^{2p-1-i}(X; \mathbb{C})/\mathcal{F}^p H^{2p-1-i}(X; \mathbb{C}) \\ \downarrow & & \downarrow \\ \pi_i(\mathbf{K}_{\mathbb{C}}(X)) & \xrightarrow{\mathbf{r}_{\mathbb{C}, \mathbf{D}, p}^{\text{Beil}}} & H^{2p-i}(\mathbf{D}(p)(X)) \end{array}$$

commutes.

We will now give the details of the constructions and a proof of this theorem using the techniques developed in the present paper. The following constructions are similar to those exposed in more detail in Subsections 4.4, 4.5 and it might be easier for the reader to skip ahead to Section 4 and to go through the rest of this subsection afterwards.

We begin by describing the ∞ -categorical version of topological K -theory. We denote by $\mathbf{Mf}_{\mathbb{C}}$ the category of complex manifolds considered as a site with the classical topology and let $\mathbf{S}^{top} := \mathbf{N}(\mathbf{Mf}_{\mathbb{C}}^{op})$ be its nerve considered as an ∞ -category. We consider the stack $i\mathbf{Vect}^{top}$ on $\mathbf{Mf}_{\mathbb{C}}$ which associates to $X \in \mathbf{Mf}_{\mathbb{C}}$ the symmetric monoidal groupoid of smooth vector bundles on X . We consider

$$i\mathbf{Vect}^{top} \in \mathbf{Fun}(\mathbf{S}^{top}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}])) .$$

This stack does not model topological K -theory since it does not take the topology of the morphism spaces into account. We can remedy this fact by an application of the functor $\bar{\mathbf{s}}$ defined in (124). We define (cf. Definition 2.2) the sheaf of topological K -theory spectra

$$\mathbf{K}^{top} := \bar{\mathbf{s}}(L(K(i\mathbf{Vect}^{top}))) \in \mathbf{Fun}^{desc}(\mathbf{S}^{top}, \mathbf{N}(\mathbf{Sp})[W^{-1}]) , \quad (36)$$

where again L denotes the sheafification functor (122). Note that, equivalently,

$$\mathbf{K}^{top} = L \circ \mathbf{sp} \circ \Omega B \circ \bar{\mathbf{s}}(\mathbf{N}(i\mathbf{Vect}^{top})) .$$

Then $\pi_i(\mathbf{K}^{top}(X)) \cong \mathbf{ku}^{-i}(X)$ is the connective topological K -theory of X .

Next, we construct the aforementioned comparison map c . Let $h: \mathbf{S}_{\mathbb{C}} \rightarrow \mathbf{S}^{top}$ be the map given by sending a smooth complex variety to its associated complex manifold. It is compatible with the topologies. We have a natural transformation $u: i\mathbf{Vect}_{\mathbb{C}} \rightarrow h^*i\mathbf{Vect}^{top}$ which maps a vector bundle to its underlying smooth complex vector bundle. The comparison map is now defined by

$$c: \mathbf{K}_{\mathbb{C}} = L(K(i\mathbf{Vect}_{\mathbb{C}})) \xrightarrow{u} h^*L(K(i\mathbf{Vect}^{top})) \xrightarrow{(125)} h^*\bar{s}(L(K(i\mathbf{Vect}^{top}))) = h^*\mathbf{K}^{top}.$$

We define the relative K -theory \mathbf{K}^{rel} by taking its fibre:

$$\mathbf{K}^{rel} \rightarrow \mathbf{K}_{\mathbb{C}} \xrightarrow{c} h^*\mathbf{K}^{top}.$$

The relative Chern character will be induced by compatible regulator maps on $\mathbf{K}_{\mathbb{C}}$ and $h^*\mathbf{K}^{top}$. We now describe the relevant choice of geometries and characteristic cocycles. A geometric smooth bundle is a pair (V, g) of a smooth bundle V together with a geometry $g = (h^V, \nabla^V)$ consisting of a hermitean metric h^V and a connection ∇^V . We let $i\mathbf{Vect}^{top, geom}$ be the symmetric monoidal stack of geometric bundles. Similar as in Lemma 4.16 the forgetful transformation induces an equivalence

$$\bar{s}(\mathbf{N}(i\mathbf{Vect}^{top, geom})) \xrightarrow{\sim} \bar{s}(\mathbf{N}(i\mathbf{Vect}^{top})). \quad (37)$$

We define the sheaf of complexes

$$\mathbf{A} := \prod_{p \geq 0} A[2p]$$

on $\mathbf{Mf}_{\mathbb{C}}$ where A is the usual de Rham complex of smooth \mathbb{C} -valued forms. We further define the characteristic $Z^0(\mathbf{A})$ -valued cocycle

$$\omega^{top}: \pi_0(i\mathbf{Vect}^{top, geom}) \rightarrow Z^0(\mathbf{A})$$

which associates to $(V, (h^V, \nabla^V)) \in i\mathbf{Vect}^{top, geom}(X)$ the form

$$\omega^{top}(V, (h^V, \nabla^V)) := \prod_{p \geq 0} \mathbf{ch}_{2p}(\nabla^V)$$

(see Subsection 4.4 for the definition of \mathbf{ch}_{2p}).

Similarly to the construction of Beilinson's regulator in Definition 4.20 we define the usual Chern character

$$\mathbf{ch}: \mathbf{K}^{top} \rightarrow H(\mathbf{A})$$

through the diagram

$$\begin{array}{ccc} \bar{s}(K(i\mathbf{Vect}_{\mathbb{C}}^{top})) & \xleftarrow[(37)]{\sim} & \bar{s}(K(i\mathbf{Vect}_{\mathbb{C}}^{top, geom})) \xrightarrow{\bar{s}K(\omega^{top})} \bar{s}(H(\mathbf{A})) \\ \downarrow (36) & & \uparrow \cong \text{Lemma A.3} \\ \mathbf{K}^{top} & \xrightarrow{\mathbf{ch}} & H(\mathbf{A}). \end{array}$$

We can also consider its real variant by replacing \mathbf{A} with the complex of real valued differential forms

$$\mathbf{A}_{\mathbb{R}} := \prod_{p \geq 0} (2\pi i)^p A_{\mathbb{R}}[2p]$$

and using the characteristic $\mathbf{A}_{\mathbb{R}}$ -valued cocycle $\omega_{\mathbb{R}}^{top} : (V, (h^V, \nabla^V)) \mapsto \prod_{p \geq 0} \mathbf{ch}_{2p}(\nabla^{V,u})$ (cf. (50)).

We now come to the construction of the regulator on the algebraic K -theory. Geometries on algebraic vector bundles are defined in such a way that the corresponding Chern character forms live in the appropriate step of the Hodge filtration. We thus consider the complex of presheaves on $\mathbf{Sm}_{\mathbb{C}}$

$$\mathcal{F}^0 \mathbf{A}_{\log} := \prod_{p \geq 0} \mathcal{F}^p A_{\log}[2p] .$$

There is a natural map $\mathcal{F}^0 \mathbf{A}_{\log} \rightarrow h^* \mathbf{A}$ which gives rise to a fibre sequence

$$\mathrm{Cone}(\mathcal{F}^0 \mathbf{A}_{\log} \rightarrow h^* \mathbf{A})[-1] \rightarrow \mathcal{F}^0 \mathbf{A}_{\log} \rightarrow h^* \mathbf{A} .$$

From a good geometry $g^V = (h^V, \nabla^{II})$ (see 4.3) we obtain a geometry (h^V, ∇^V) with $\nabla^V = \nabla^{II} + \nabla^I + \bar{\partial}$ on the underlying smooth bundle.

The characteristic form

$$\omega_{\mathcal{F}}(g^V) := \prod_{p \geq 0} \mathbf{ch}_{2p}(\nabla^V)$$

defines a characteristic cocycle

$$\omega_{\mathcal{F}} : \pi_0(i\mathbf{Vect}_{\mathbb{C}}^{geom}) \rightarrow Z^0(\mathcal{F}^0 \mathbf{A}_{\log})$$

and again similar to the construction in Definition 4.20 a regulator map

$$\mathbf{ch}_{\mathcal{F}} : \mathbf{K}_{\mathbb{C}} \rightarrow H(\mathcal{F}^0 \mathbf{A}_{\log}) .$$

Clearly, the diagram of characteristic cocycles

$$\begin{array}{ccc} \pi_0(i\mathbf{Vect}_{\mathbb{C}}^{geom}) & \xrightarrow{\omega_{\mathcal{F}}} & Z^0(\mathcal{F}^0 \mathbf{A}_{\log}) \\ \downarrow & & \downarrow \\ h^* \pi_0(\mathbf{Vect}_{\mathbb{C}}^{top, geom}) & \xrightarrow{\omega^{top}} & h^* Z^0(\mathbf{A}) \end{array}$$

commutes. We thus get the commutativity of the lower square in the following diagram and define the relative Chern character \mathbf{ch}^{rel} to be the map induced on fibres:

$$\begin{array}{ccc} \mathbf{K}^{rel} & \xrightarrow{\mathbf{ch}^{rel}} & H(\mathrm{Cone}(\mathcal{F}^0 \mathbf{A}_{\log} \rightarrow h^* \mathbf{A})[-1]) \\ \downarrow & & \downarrow \\ \mathbf{K}_{\mathbb{C}} & \xrightarrow{\mathbf{ch}_{\mathcal{F}}} & H(\mathcal{F}^0 \mathbf{A}_{\log}) \\ \downarrow & & \downarrow \\ h^* \mathbf{K}^{top} & \xrightarrow{\mathbf{ch}} & H(h^* \mathbf{A}) . \end{array}$$

Note that we have natural isomorphisms

$$\pi_i \left(H(\text{Cone}(\mathcal{F}^0 \mathbf{A}_{\log} \rightarrow h^* \mathbf{A})[-1])(X) \right) \cong \prod_{p \geq 0} H^{2p-1-i}(X; \mathbb{C}) / \mathcal{F}^p H^{2p-1-i}(X; \mathbb{C}).$$

This finishes the construction of the relative Chern character.

In order to compare it with the Beilinson regulator we introduce the *Deligne-Beilinson* or *weak absolute Hodge complex*

$$\mathbf{D}(p) := \text{Cone}((2\pi i)^p A_{\mathbb{R}} \oplus \mathcal{F}^p A_{\log} \rightarrow A)[2p-1] \in \mathbf{PSh}_{\text{Ch}}(\mathbf{Sm}_{\mathbb{C}}) \quad (38)$$

and $\mathbf{D} := \prod_{p \geq 0} \mathbf{D}(p)$. Note that we have a natural map $\mathbf{DR}_{\mathbb{C}} \rightarrow \mathbf{D}$ by simply forgetting the weight filtration and the logarithmic growth condition at the appropriate places.

Upon restriction along $\mathbf{S}_{\mathbb{C}} \rightarrow \mathbf{S}_{Mf, \mathbb{C}}$ and composition with the map $\mathbf{DR}_{\mathbb{C}} \rightarrow \mathbf{D}$ the characteristic $\mathbf{DR}_{\mathbb{C}}$ -valued cocycle ω defined in (59) gives a \mathbf{D} -valued characteristic cocycle which we denote by $\omega_{\mathbf{D}}$. The induced regulator

$$\mathbf{r}_{\mathbb{C}, \mathbf{D}}^{\text{Beil}} : \mathbf{K}_{\mathbb{C}} \rightarrow H(\mathbf{D}) \quad (39)$$

is Beilinson's regulator for complex algebraic varieties. It follows from the constructions that the composition of $\omega_{\mathbf{D}}$ with $Z^0(\mathbf{D}) \rightarrow Z^0(\mathcal{F}^0 \mathbf{A}_{\log})$ coincides with $\omega_{\mathcal{F}}$ and moreover that the square

$$\begin{array}{ccc} \pi_0(i\text{Vect}_{\mathbb{C}}^{\text{geom}}) & \xrightarrow{\omega_{\mathbf{D}}} & Z^0(\mathbf{D}) \\ \downarrow & & \downarrow \\ h^* \pi_0(i\text{Vect}^{\text{top, geom}}) & \xrightarrow{\omega_{\mathbb{R}}^{\text{top}}} & Z^0(h^* \mathbf{A}_{\mathbb{R}}) \end{array}$$

commutes.

We now use the fact that the fibre of the map $\mathbf{D} \rightarrow h^* \mathbf{A}_{\mathbb{R}}$ is naturally equivalent to $\text{Cone}(\mathcal{F}^0 \mathbf{A}_{\log} \rightarrow h^* \mathbf{A})[-1]$, and hence that $\mathbf{r}_{\mathbb{C}, \mathbf{D}}^{\text{Beil}}$ also induces a map

$$\mathbf{K}^{\text{rel}} \rightarrow H(\text{Cone}(\mathcal{F}^0 \mathbf{A}_{\log} \rightarrow h^* \mathbf{A})[-1]). \quad (40)$$

More precisely, we have the following commutative diagram

$$\begin{array}{ccccc} \text{Cone}(\mathbf{D} \rightarrow h^* \mathbf{A}_{\mathbb{R}})[-1] & \longrightarrow & \mathbf{D} & \longrightarrow & h^* \mathbf{A}_{\mathbb{R}} \\ \uparrow \sim \text{induced} & & \parallel & & \uparrow \sim \\ \text{Cone}(\mathbf{D} \rightarrow \text{Cone}(\cdots))[-1] & \longrightarrow & \mathbf{D} & \longrightarrow & \text{Cone}(h^* \mathbf{A}_{\mathbb{R}} \oplus h^* \mathbf{A} \rightarrow h^* \mathbf{A})[-1] \\ \downarrow \sim \text{induced} & & \downarrow & & \downarrow \\ \text{Cone}(\mathcal{F}^0 \mathbf{A}_{\log} \rightarrow h^* \mathbf{A})[-1] & \longrightarrow & \mathcal{F}^0 \mathbf{A}_{\log} & \longrightarrow & h^* \mathbf{A}. \end{array} \quad (41)$$

Here the map from \mathbf{D} to the cone in the right column is induced by $\mathcal{F}^0 \mathbf{A}_{\log} \rightarrow h^* \mathbf{A}$, the maps in the right column are the obvious projections, and the maps in the left column

are the induced ones. The lower left vertical arrow is an equivalence since the lower right square is a pullback square.

We claim that (40) coincides with \mathbf{ch}^{rel} constructed above. The assertion of Theorem 3.10 then follows by taking homotopy groups. To prove the claim, we introduce the auxiliary characteristic cocycle

$$\omega_{\mathbf{Cone}} : \pi_0(i\mathbf{Vect}^{top, geom}) \rightarrow Z^0(\mathbf{Cone}(\mathbf{A}_{\mathbb{R}} \oplus \mathbf{A} \rightarrow \mathbf{A})[-1])$$

which sends $(V, (h^V, \nabla^V))$ to $\prod_{p \geq 0} (\mathbf{ch}_{2p}(\nabla^{V,u}) \oplus \mathbf{ch}_{2p}(\nabla^V), \widetilde{\mathbf{ch}}_{2p-1}(\nabla^{V,u}, \nabla^V))$ (see Subsection 4.4 for the definition of the transgression $\widetilde{\mathbf{ch}}$ and compare with (53)). The claim then follows from the compatibility of characteristic cocycles defined on $\pi_0(i\mathbf{Vect}_{\mathbb{C}}^{geom})$ respectively $h^*\pi_0(i\mathbf{Vect}^{top, geom})$ with values in the groups of 0-cocycles in the middle respectively right column of diagram (41) summarized schematically in the diagram

$$\begin{array}{ccc} \omega_{\mathbf{D}} & \xrightarrow{\quad} & \omega_{\mathbb{R}}^{top} \\ \parallel & & \uparrow \\ \omega_{\mathbf{D}} & \xrightarrow{\quad} & \omega_{\mathbf{Cone}} \\ \downarrow & & \downarrow \\ \omega_{\mathcal{F}} & \xrightarrow{\quad} & \omega^{top}. \end{array}$$

This compatibility in turn follows directly from the constructions. \square

Remark 3.11. *Whereas with this construction of the relative Chern character the relation to Beilinson's regulator follows quite easily, it is not a priori clear that this new construction really extends Karoubi's original one. This is done in [Tam12] and used there to establish also the comparison with Borel's regulator in the number field case.*

4 Smooth manifolds

In this Section we provide the necessary geometric background for the constructions of Section 3 in the more general setting where we consider products $M \times X$ of an arbitrary smooth manifold M and an algebraic variety (or regular, separated scheme of finite type over \mathbb{Z}) X . In 4.2 we extend the definition of Burgos' complex [Bur94] to this situation and use it to define the absolute Hodge complex $\mathbf{DR}_{Mf}(M \times X)$. In 4.3 we introduce the notion of a good geometry and the characteristic form associated with a good geometry in the absolute Hodge complex. This Subsection also contains the proof of the important fact that the space of good geometries is contractible. Using the extension of Beilinson's regulator map to the case $M \times X$ we define in Section 5 the differential algebraic K -theory spectrum, study its homotopy groups, and construct the geometric cycle map.

4.1 The sites

We consider the category \mathbf{Mf} of smooth manifolds as a site equipped with the topology of open coverings. The nerve of its opposite is the ∞ -category $\mathbf{S}_{Mf} := \mathbf{N}(\mathbf{Mf}^{op})$.

We let $\mathbf{Sm}_{\mathbb{C}}$ denote the site of smooth varieties over \mathbb{C} (see 3.1) and Zariski open coverings and set $\mathbf{S}_{\mathbb{C}} := \mathbf{N}(\mathbf{Sm}_{\mathbb{C}}^{op})$.

Finally, we let $\mathbf{Reg}_{\mathbb{Z}}$ be the site of regular separated schemes of finite type over $\mathbf{Spec}(\mathbb{Z})$ (see 3.2) with the topology of Zariski open coverings and write $\mathbf{S}_{\mathbb{Z}} := \mathbf{N}(\mathbf{Reg}_{\mathbb{Z}}^{op})$.

We consider the product sites

$$\mathbf{Mf} \times \mathbf{Sm}_{\mathbb{C}} , \quad \mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$$

and write

$$\mathbf{S}_{Mf, \mathbb{C}} := \mathbf{S}_{Mf} \times \mathbf{S}_{\mathbb{C}} , \quad \mathbf{S}_{Mf, \mathbb{Z}} := \mathbf{S}_{Mf} \times \mathbf{S}_{\mathbb{Z}}$$

for the corresponding ∞ -categories. We write objects of these product sites in the form $M \times X$, where $M \in \mathbf{Mf}$ and X is the algebraic object. We have canonical topology preserving functors

$$e : \mathbf{S}_{\mathbb{C}} \rightarrow \mathbf{S}_{Mf, \mathbb{C}} , \quad \mathbf{S}_{\mathbb{Z}} \rightarrow \mathbf{S}_{Mf, \mathbb{Z}} \quad (42)$$

both induced by $X \mapsto * \times X$, where $*$ is the point considered as a manifold.

4.2 The sheaves $\mathbf{DR}_{Mf, \mathbb{C}}$ and $\mathbf{DR}_{Mf, \mathbb{Z}}$

We start with the construction of the analogue

$$A_{\log} \in \mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf, \mathbb{C}}, \mathbf{N}(\mathbf{Ch})[W^{-1}])$$

of the bifiltered complex introduced in Definition 3.1. The meaning of the constancy condition is explained in Subsection A.9. For a smooth manifold N we let $A(N)_{\mathbb{R}} \subseteq A(N)$ denote the smooth de Rham complex and its subcomplex of real forms. Let M be a manifold and X a smooth variety over \mathbb{C} (cf. Subsection 3.1). Then we fix some good compactification $X \xrightarrow{j} \overline{X}$ and write $D := \overline{X} - X$. We consider X and \overline{X} as complex manifolds and define the real dg-algebra

$$A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D) \subseteq A(M \times X)_{\mathbb{R}}$$

to be the sub- $A(M \times \overline{X})_{\mathbb{R}}$ -algebra which is locally generated by 1 and

$$\log(z_i \bar{z}_i), \quad \operatorname{Re} \frac{dz_i}{z_i}, \quad \operatorname{Im} \frac{dz_i}{z_i}, \quad \text{for } i \in I . \quad (43)$$

Here the z_i , $i \in I$, are local coordinates of \overline{X} (for the analytic topology) which define D locally by the equation $\prod_{i \in I} z_i = 0$.

The following example should clarify the meaning of the notion *locally generated*. We consider $X := \mathbb{CP}^1 \setminus \{p\}$ with local coordinate z at p and $M = \mathbb{R}$. We let $(\chi_n)_{n \in \mathbb{Z}}$ be a partition of unity on \mathbb{R} such that $\text{supp}(\chi_n) \subset [n-2, n+2]$ for all $n \in \mathbb{Z}$. Then

$$\sum_{n \in \mathbb{Z}} \log(|z|)^{|n|} \chi_n \in A_{\mathbb{R} \times \mathbb{CP}^1}^0(\mathbb{R} \times X, \log \{p\}) . \quad (44)$$

We are going to introduce several filtrations on $A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D)$. The naive weight filtration $\widetilde{\mathcal{W}}$ is the multiplicative increasing filtration of $A(M \times \overline{X})_{\mathbb{R}}$ -modules obtained by assigning weight 0 to the section 1 and weight 1 to the sections listed in (43).

We define a decreasing filtration \mathcal{L} of $A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D)$ such that $\mathcal{L}^p A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D)$ is the subcomplex of differential forms that are given locally by

$$\sum_{I, J, K, |I| \geq p} \omega_{I, J, K} dx^I \wedge \text{Re } dz^J \wedge \text{Im } dz^K ,$$

where the x_i are local coordinates for M , the z_j local coordinates for \overline{X} , and $\omega_{I, J, K}$ local smooth functions on $M \times X$.

We now define the weight filtration \mathcal{W} as the diagonal filtration of $\widetilde{\mathcal{W}}$ and \mathcal{L} :

$$\mathcal{W}_k A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D) := \sum_p \widetilde{\mathcal{W}}_{k+p} \cap \mathcal{L}^p A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D).$$

As usual, its décalage (cf. (16)) will be denoted by $\hat{\mathcal{W}}_* A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D)$.

We further define the complex dg-algebra

$$A_{M \times \overline{X}}(M \times X, \log D) := A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D) \otimes_{\mathbb{R}} \mathbb{C}$$

with the induced weight filtration. This complex carries the decreasing Hodge filtration \mathcal{F} such that the elements of $\mathcal{F}^p A_{M \times \overline{X}}(M \times X, \log D)$ are locally of the form

$$\sum_{I, J, K, |J| \geq p} \omega_{I, J, K} dx^I \wedge dz^J \wedge d\bar{z}^K ,$$

where the x_i, z_j are local coordinates of M and X , respectively.

In the special case that $M = *$ the complex $A_{* \times \overline{X}, \mathbb{R}}(* \times X, \log D)$ is exactly the complex of global sections of Burgos' sheaf of complexes with its filtrations [Bur94, Section 2], (14).

Fix $X \hookrightarrow \overline{X}$ as above. Then the functors

$$\begin{aligned} M &\mapsto A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D) , \\ M &\mapsto \hat{\mathcal{W}}_k A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D) \\ M &\mapsto A_{M \times \overline{X}}(M \times X, \log D) \\ M &\mapsto \hat{\mathcal{W}}_k \cap \mathcal{F}^p A_{M \times \overline{X}}(M \times X, \log D) \end{aligned} \quad (45)$$

will be considered as objects of $\mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Mf})$. They are sub-presheaves of the presheaf $M \mapsto A(M \times X)$.

Lemma 4.1. *The presheaves of complexes (45) satisfy descent and are constant in the sense of Definition A.4, i.e. they belong to $\mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf}, \mathbf{N}(\mathbf{Ch})[W^{-1}])$.*

Proof. In order to verify descent we show below that the presheaves of complexes (45) are degree-wise sheaves of modules over the sheaf $C_{\mathbb{R}}^{\infty} \in \mathbf{Sh}_{\mathbf{Alg}(\mathbb{R})}(\mathbf{Mf})$ of algebras of smooth real-valued functions. Since the augmented Čech complex of a sheaf of $C_{\mathbb{R}}^{\infty}$ -modules with respect to any open covering is exact it follows that they belong to $\mathbf{Fun}^{desc}(\mathbf{S}_{Mf}, \mathbf{N}(\mathbf{Ch})[W^{-1}])$. We discuss the case of $\hat{\mathcal{W}}_k A_{\dots \times \overline{X}, \mathbb{R}}(\dots \times X, \log D)$. The other cases are similar.

As a preparation note that for an n -form $\omega \in \mathcal{W}_{k-n}$ we have $\omega \in \hat{\mathcal{W}}_k$ if and only if $d^X \omega \in \mathcal{W}_{k-n-1}$, where d^X is the differential in X -direction. Indeed, we may assume that $\omega \in \widetilde{\mathcal{W}}_{k-n+p} \cap \mathcal{L}^p$ for some p . Then $d\omega = d^M \omega \pm d^X \omega$ and

$$d^M \omega \in \widetilde{\mathcal{W}}_{k-n+p} \cap \mathcal{L}^{p+1} \subseteq \mathcal{W}_{k-n-1} .$$

Now we assume that

$$\omega \in \hat{\mathcal{W}}_k A_{M \times \overline{X}, \mathbb{R}}^n(M \times X, \log D) , \quad f \in C^{\infty}(M, \mathbb{R}) .$$

Then obviously $f\omega \in \mathcal{W}_{k-n}$ and $d^X(f\omega) = f d^X \omega \in \mathcal{W}_{k-n-1}$, hence

$$f\omega \in \hat{\mathcal{W}}_k A_{M \times \overline{X}, \mathbb{R}}^n(M \times X, \log D) .$$

In order to show that the sheaves of complexes (45) are constant we verify homotopy invariance and use Lemma A.5. The verification of homotopy invariance is standard once we know that the integration

$$\int_I : A^*(I \times M \times X) \rightarrow A^{*-1}(M \times X)$$

preserves the subcomplexes (45).

First note that \int_I preserves the subcomplex of forms with logarithmic singularities along D . For the filtrations we only discuss the case $\hat{\mathcal{W}}_k A_{\dots \times \overline{X}, \mathbb{R}}(\dots \times X, \log D)$. Take

$$\omega \in \hat{\mathcal{W}}_k A_{I \times M \times \overline{X}, \mathbb{R}}^n(I \times M \times X, \log D) .$$

We may again assume that $\omega \in \widetilde{\mathcal{W}}_{k-n+p} \cap \mathcal{L}^p$ for some p . Then

$$\int_I \omega \in \widetilde{\mathcal{W}}_{k-n+p} \cap \mathcal{L}^{p-1} \subseteq \mathcal{W}_{k-n+1}$$

and $d^X \int_I \omega = - \int_I d^X \omega \in \mathcal{W}_{k-n}$ since $d^X \omega \in \mathcal{W}_{k-n-1}$. This shows that

$$\int_I \omega \in \hat{\mathcal{W}}_k A_{M \times \overline{X}, \mathbb{R}}^{n-1}(M \times X, \log D) .$$

□

Now assume that we have two good compactifications \overline{X} and \overline{X}' of X and a morphism $\overline{X}' \rightarrow \overline{X}$ inducing the identity on X . Then the induced maps

$$\begin{aligned} A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D) &\rightarrow A_{M \times \overline{X}', \mathbb{R}}(M \times X, \log D') , \\ \hat{\mathcal{W}}_k A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log D) &\rightarrow \hat{\mathcal{W}}_k A_{M \times \overline{X}', \mathbb{R}}(M \times X, \log D') , \\ \mathcal{F}^p A_{M \times \overline{X}}(M \times X, \log D) &\rightarrow \mathcal{F}^p A_{M \times \overline{X}'}(M \times X, \log D') , \\ \hat{\mathcal{W}}_k \cap \mathcal{F}^p A_{M \times \overline{X}}(M \times X, \log D) &\rightarrow \hat{\mathcal{W}}_k \cap \mathcal{F}^p A_{M \times \overline{X}'}(M \times X, \log D') \end{aligned}$$

are quasi-isomorphisms. Indeed, descent and constancy in M reduce the claim to the case $M = *$ and there the claim follows from Hodge theory (cf. the discussion following (18)). In order to get rid of the choice of the compactification \overline{X} we now proceed as in Subsection 3.1. We define the presheaf $A_{\log, Mf, \mathbb{R}} \in \mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Mf} \times \mathbf{Sm}_{\mathbb{C}})$ by

$$A_{\log, Mf, \mathbb{R}}(M \times X) := \operatorname{colim}_{\overline{X} \in I_X} A_{M \times \overline{X}, \mathbb{R}}(M \times X, \log(\overline{X} - X)) \quad (46)$$

where the colimit runs over the directed system I_X of all good compactifications of X . It has an induced weight filtration \mathcal{W} and a Hodge filtration \mathcal{F} on the complexification

$$A_{\log, Mf}(M \times X) := \operatorname{colim}_{\overline{X} \in I_X} A_{M \times \overline{X}}(M \times X, \log(\overline{X} - X)) \cong A_{\log, Mf, \mathbb{R}}(M \times X) \otimes_{\mathbb{R}} \mathbb{C} .$$

By the above, all maps in the directed system are quasi-isomorphisms, which are bifiltered with respect to the décalage $\hat{\mathcal{W}}$ of the weight filtration and the Hodge filtration.

Lemma 4.2. *We have*

$$A_{\log, Mf} , A_{\log, Mf, \mathbb{R}} , \hat{\mathcal{W}}_k A_{\log, Mf, \mathbb{R}} , \mathcal{F}^p \cap \hat{\mathcal{W}}_k A_{\log, Mf} \in \mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf, \mathbb{C}}, \mathbf{N}(\mathbf{Ch})[W^{-1}]) .$$

Proof. We can check descent in the M - and X -directions separately. Since the structure maps of the colimit over I_X are quasi-isomorphisms it follows from Lemma 4.1 that the complexes in the statement of the Lemma fulfill descent and constancy in the M -direction. In order to show Zariski descent in the X -direction we use descent and constancy in the M -direction in order to reduce to the case $M = *$. In this special case the Zariski descent was proven as Lemma 3.2. \square

Definition 4.3. *For $p \geq 0$ we define the complex*

$$\mathbf{DR}_{Mf, \mathbb{C}}(p) := \operatorname{Cone}((2\pi i)^p \hat{\mathcal{W}}_{2p} A_{\log, Mf, \mathbb{R}} \oplus \hat{\mathcal{W}}_{2p} \cap \mathcal{F}^p A_{\log, Mf} \xrightarrow{(\alpha, \beta) \mapsto \alpha - \beta} \hat{\mathcal{W}}_{2p} A_{\log, Mf})[2p-1] .$$

and

$$\mathbf{DR}_{Mf, \mathbb{C}} := \prod_{p \geq 0} \mathbf{DR}_{Mf, \mathbb{C}}(p) .$$

By the above, we may view $\mathbf{DR}_{Mf, \mathbb{C}}(p)$ and $\mathbf{DR}_{Mf, \mathbb{C}}$ as objects in $\mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Mf} \times \mathbf{Sm}_{\mathbb{C}})$ or in $\mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf, \mathbb{C}}, \mathbf{N}(\mathbf{Ch})[W^{-1}])$. Furthermore, by the constancy of $\mathbf{DR}_{Mf, \mathbb{C}}$ and using that $\mathbf{DR}_{\mathbb{C}} \cong e^* \mathbf{DR}_{Mf, \mathbb{C}}$ we have an equivalence

$$L(\operatorname{pr}^* \mathbf{DR}_{\mathbb{C}}) \xrightarrow{\sim} \mathbf{DR}_{Mf, \mathbb{C}} \quad (47)$$

in $\mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf, \mathbb{C}}, \mathbf{N}(\mathbf{Ch})[W^{-1}])$, where $\operatorname{pr} : \mathbf{S}_{Mf, \mathbb{C}} \rightarrow \mathbf{S}_{\mathbb{C}}$ denotes the projection.

Definition 4.4. Generalizing Definition 3.4 we define

$$\mathbf{DR}_{Mf,\mathbb{Z}} \in \mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}})$$

by

$$\mathbf{DR}_{Mf,\mathbb{Z}} := ((\mathrm{id}_{\mathbf{S}_{Mf}} \times B)^* \mathbf{DR}_{Mf,\mathbb{C}})^{\mathbb{Z}/2\mathbb{Z}} .$$

We have

$$\mathbf{DR}_{Mf,\mathbb{Z}} \in \mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf,\mathbb{Z}}, \mathbf{N}(\mathbf{Ch})[W^{-1}]) .$$

In order to understand the cohomology of $\mathbf{DR}_{Mf,\mathbb{C}}$ we show a version of the de Rham Lemma. Recall the Eilenberg-MacLane correspondence H (see A.5) and the smooth function object $\mathbf{Sm}(\dots)$ introduced in A.10.

Lemma 4.5. *We have canonical equivalences*

$$H(\mathbf{DR}_{Mf,\mathbb{C}}) \cong \mathbf{Sm}(H(\mathbf{DR}_{\mathbb{C}})) , \quad H(\mathbf{DR}_{Mf,\mathbb{Z}}) \cong \mathbf{Sm}(H(\mathbf{DR}_{\mathbb{Z}})) . \quad (48)$$

Proof. We consider the complex case. The arithmetic case is similar. By Lemma A.7 we have a diagram of canonical morphisms

$$H(\mathbf{DR}_{Mf,\mathbb{C}}) \xleftarrow{\sim} L(\mathrm{pr}^* e^* H(\mathbf{DR}_{Mf,\mathbb{C}})) \cong L(\mathrm{pr}^* e^* \mathbf{Sm}(H(\mathbf{DR}_{\mathbb{C}}))) \xrightarrow{\sim} \mathbf{Sm}(H(\mathbf{DR}_{\mathbb{C}})) .$$

□

Using (129) we get the following consequence.

Corollary 4.6. *We have isomorphisms*

$$H^*(\mathbf{DR}_{Mf,\mathbb{C}}(M \times X)) \cong H(\mathbf{DR}_{\mathbb{C}}(X))^*(M) , \quad X \in \mathbf{Sm}_{\mathbb{C}}$$

and

$$H^*(\mathbf{DR}_{Mf,\mathbb{Z}}(M \times X)) \cong H(\mathbf{DR}_{\mathbb{Z}}(X))^*(M) , \quad X \in \mathbf{Reg}_{\mathbb{Z}} .$$

In other words, $H^*(\mathbf{DR}_{Mf,\mathbb{C}}(M \times X))$ is the cohomology of M with coefficients in the absolute Hodge cohomology of X .

4.3 Vector bundles and geometries

We consider a product $M \times X$ of a smooth (real) manifold M and a smooth complex manifold X . We let $V \rightarrow M \times X$ be a complex vector bundle.

Definition 4.7. *A partial geometry on V is a pair $(\nabla^I, \bar{\partial})$ consisting of*

1. *a partial connection ∇^I in the M -direction,*
2. *a holomorphic structure $\bar{\partial}$ in the X -direction.*

Definition 4.8. *A geometry on the vector bundle V with a partial geometry $(\nabla^I, \bar{\partial})$ is a pair $g^V := (h^V, \nabla^H)$ consisting of*

1. a partial connection ∇^I in the X -direction which extends the holomorphic structure $\bar{\partial}$, and
2. a hermitean metric h^V on the bundle $V \rightarrow M \times X$.

We now assume that X is a smooth variety over \mathbb{C} and consider a sheaf V of locally free finitely generated $\mathbf{pr}_X^* \mathcal{O}_X$ -modules on $M \times X$ where \mathbf{pr}_X^* denotes the inverse image under projection to X in the sense of sheaves of sets. We use the same symbol V in order to denote the corresponding complex vector bundle over $M \times X$. It has an induced partial geometry $(\nabla^I, \bar{\partial})$ which in addition satisfies:

Assumption 4.9. 1. The partial connection ∇^I is flat.

2. The holomorphic structure $\bar{\partial}$ is constant w.r.t. ∇^I , i.e. $[\nabla^I, \bar{\partial}] = 0$.

The original locally free sheaf of $\mathbf{pr}_X^* \mathcal{O}_X$ -modules is the sheaf of sections annihilated by both, the connection ∇^I and the holomorphic structure $\bar{\partial}$.

In Definition 4.13 we will introduce characteristic forms associated to geometries. If X is not proper over \mathbb{C} we want these characteristic forms to belong to the logarithmic subcomplex $A_{\log, Mf}(M \times X)$ with controlled weights. To this end we define a subset of geometries on V called good geometries which behave in a controlled way at infinity. The definition and the construction in the following lemma are inspired by the constructions of Burgos and Wang in [BW98, §§2,3].

Definition 4.10. A geometry g on V is called good if every $m \in M$ has a neighbourhood $U \subseteq M$ such that there exist the following data:

1. a smooth proper variety Y over \mathbb{C} ,
2. a good compactification $Y \times X \hookrightarrow \overline{Y \times X}$ (see 3.1),
3. a locally free sheaf of $\mathbf{pr}_{23}^* \mathcal{O}_{\overline{Y \times X}}$ -modules \bar{V} on $U \times \overline{Y \times X}$, where $\mathbf{pr}_{23} : U \times \overline{Y \times X} \rightarrow \overline{Y \times X}$ is the projection,
4. a natural map of $\mathbf{pr}_3^* \mathcal{O}_X$ -modules $\mathbf{pr}_{13}^* V|_{U \times X} \rightarrow \bar{V}|_{U \times Y \times X}$ which induces an isomorphism $\mathbf{pr}_{13}^* V|_{U \times X} \otimes_{\mathbf{pr}_3^* \mathcal{O}_X} \mathbf{pr}_{23}^* \mathcal{O}_{Y \times X} \cong \bar{V}|_{U \times Y \times X}$, where $\mathbf{pr}_{13} : U \times Y \times X \rightarrow U \times X$ and $\mathbf{pr}_3 : U \times Y \times X \rightarrow X$ again denote the projections,
5. a geometry \bar{g} on \bar{V} in the sense of Definition 4.8 (note that \bar{V} has a natural flat partial connection in the M -direction and a holomorphic structure along $\overline{Y \times X}$),
6. a smooth map $\lambda : U \rightarrow Y$ such that $g|_{U \times X} = \Phi_\lambda^* \bar{g}$, where

$$\Phi_\lambda : U \times X \rightarrow U \times Y \times X, \quad (u, x) \mapsto (u, \lambda(u), x) \quad (49)$$

and we use the isomorphism $\Phi_\lambda^* \bar{V} \cong V|_{U \times X} \otimes_{\mathbf{pr}_X^* \mathcal{O}_X} \Phi_\lambda^* \mathbf{pr}_{23}^* \mathcal{O}_{Y \times X}$ which is naturally induced by 4.

Definition 4.11. A sheaf V of locally free finitely generated $\mathbf{pr}_X^* \mathcal{O}_X$ -modules on $M \times X$ together with a good geometry g will be called a *geometric bundle* and denoted by (V, g) .

Lemma 4.12. *Good geometries exist. A good geometry defined over $\partial \Delta^n \times M \times X$ can be extended to a good geometry on $\Delta^n \times M \times X$.*

Proof. First observe that good geometries exist locally on M . For instance, if M is simply connected the sheaf V is of the form $\mathbf{pr}_X^* V_0$ for some locally free \mathcal{O}_X -module V_0 of finite rank. By [BW98, Proposition 2.2] V_0 has a compactification \bar{V}_0 , more precisely, there exists a good compactification \bar{X} of X together with a locally free sheaf of $\mathcal{O}_{\bar{X}}$ -modules of finite rank \bar{V}_0 such that $\bar{V}_0|_{\partial X} \cong V_0$. A geometry on \bar{V}_0 induces a good geometry on V where we take Y to be a point.

It is thus enough to show that a collection of good geometries $(g_i)_{i=0}^n$ on V over $M \times X$ with simply connected M can be glued using a partition of unity.

We assume that $g_i = (h_i, \nabla_i^{II})$ comes from a compactification \bar{V}_i with geometry \bar{g}_i on $M \times \bar{Y}_i \times \bar{X}$ for some smooth proper variety Y_i , identification $\mathbf{pr}_{13}^* V \otimes_{\mathbf{pr}_3^* \mathcal{O}_X} \mathbf{pr}_{23}^* \mathcal{O}_{Y \times X} \cong \bar{V}_i|_{M \times Y_i \times X}$ and smooth map $\lambda_i : M \rightarrow Y_i$.

We let $[x_0, \dots, x_n]$ be homogeneous coordinates on \mathbb{CP}^n which will be considered as sections $x_i \in \mathcal{O}_{\mathbb{CP}^n}(1)(\mathbb{CP}^n)$. We let

$$Y := \mathbb{CP}^n \times \prod_{i=0}^n Y_i$$

and choose a good compactification $\bar{Y} \times \bar{X}$ mapping to the compactifications $\mathbb{CP}^n \times \prod_{i=0, i \neq j}^n Y_i \times \bar{Y}_j \times \bar{X}$ for all $j = 0, \dots, n$. We define an embedding

$$\mathbf{pr}_{13}^* V \hookrightarrow \bigoplus_{i=0}^n \mathbf{pr}_{\mathbb{CP}^n}^* \mathcal{O}_{\mathbb{CP}^n}(1) \otimes_{\mathbb{C}} \mathbf{pr}_{13}^* V =: W, \quad s \mapsto \bigoplus_{i=0}^n \mathbf{pr}_{\mathbb{CP}^n}^* x_i \otimes s,$$

where $\mathbf{pr}_{13} : M \times Y \times X \rightarrow M \times X$ and $\mathbf{pr}_{\mathbb{CP}^n} : M \times Y \times X \rightarrow \mathbb{CP}^n$ are the projections. The compactification \bar{V}_i with geometry \bar{g}_i lifts to $M \times \bar{Y} \times \bar{X}$, and these lifts will be denoted by \hat{V}_i and \hat{g}_i . Since M is simply connected the compactification $\bar{W} := \bigoplus_{i=0}^n \mathbf{pr}_{\mathbb{CP}^n}^* \mathcal{O}_{\mathbb{CP}^n}(1) \otimes \hat{V}_i$ of W determines by [BW98, Thm 2.4] after possibly replacing $\bar{Y} \times \bar{X}$ by another good compactification of $Y \times X$ mapping to $\bar{Y} \times \bar{X}$ a compactification \bar{V} of $\mathbf{pr}_{13}^* V$ such that $\bar{V} \hookrightarrow \bar{W}$ is a subbundle.

Let $H_{\mathbb{CP}^n}(1) \rightarrow \mathbb{CP}^n$ denote the holomorphic vector bundle whose sheaf of holomorphic sections is $\mathcal{O}_{\mathbb{CP}^n}(1)$. We use the standard metric $h^{H_{\mathbb{CP}^n}(1)}$ and connection on $H_{\mathbb{CP}^n}(1)$ and the geometries \hat{g}_i in order to define a geometry $g_{\bar{W}}$ on \bar{W} . We obtain a metric \bar{h} on \bar{V} by restricting the metric of \bar{W} . Using the projection $\bar{W} \rightarrow \bar{V}$ given by the metric on \bar{W} we obtain a partial connection $\bar{\nabla}^{II}$ on \bar{V} . Hence we have a geometry $\bar{g} = (\bar{h}, \bar{\nabla}^{II})$ on \bar{V} .

Let now $(\chi_i)_{i=0}^n$ be smooth real valued functions such that the $(\chi_i^2)_{i=0}^n$ form a partition of unity on M . Then we define the map

$$\lambda : M \rightarrow Y, \quad \lambda(m) := ([\chi_0(m) : \dots : \chi_n(m)], \lambda_0(m), \dots, \lambda_n(m)).$$

The data $Y \times X \hookrightarrow \overline{Y \times X}, \overline{V}, \bar{g}$ and λ show that the geometry $\tilde{g} = (\tilde{h}, \tilde{\nabla}^H) := (\Phi_\lambda^* \bar{h}, \Phi_\lambda^* \bar{\nabla}^H)$ is good. We claim that this is just the affine combination $(\sum_i \chi_i^2 h_i, \sum_i \chi_i^2 \nabla_i^H)$ of the original geometries. This can be checked using the formula $h^{H_{\mathbb{CP}^n}(1)}(x_i([z_0, \dots, z_n])) = \frac{|z_i|^2}{\sum_j |z_j|^2}$, the fact that the orthogonal projection from W to $\mathbf{pr}_{13}^* V$ of an element of the form $\oplus_i \mathbf{pr}_{\mathbb{CP}^n}^* x_i \otimes s_i$ is given by $\oplus_i \mathbf{pr}_{\mathbb{CP}^n}^* x_i \otimes \left(\frac{\sum_j |x_j|^2 s_j}{\sum_j |x_j|^2} \right)$, and observing that the pullback of the connection on $H_{\mathbb{CP}^n}(1)$ along Φ_λ becomes trivial in the X -direction. \square

4.4 Characteristic forms

We again consider a smooth manifold M , a complex manifold X , and a complex vector bundle $V \rightarrow M \times X$ with partial geometry $(\nabla^I, \bar{\partial})$ (see Definition 4.7). The choice of a geometry allows us to define the connection $\nabla^V := \nabla^I + \nabla^H$, its adjoint $\nabla^{V,*}$ with respect to h^V , and the unitarization

$$\nabla^{V,u} := \frac{1}{2}(\nabla^V + \nabla^{V,*}) . \quad (50)$$

The component in degree $2p$ of the unnormalized Chern form of the unitary connection $\nabla^{V,u}$ satisfies

$$\mathbf{ch}_{2p}(\nabla^{V,u}) := \left[\mathrm{Tr} \exp(-R^{\nabla^{V,u}}) \right]_{2p} \in (2\pi i)^p A_{\mathbb{R}}^{2p}(M \times X) .$$

If the partial geometry $(\nabla^I, \bar{\partial})$ satisfies Assumption 4.9, we have

$$R^{\nabla^V} \in \mathcal{F}^1 A^2(M \times X, \mathrm{End}(V))$$

so that

$$\mathbf{ch}_{2p}(\nabla^V) \in \mathcal{F}^p A^{2p}(M \times X) .$$

Finally we consider the transgression

$$\tilde{\mathbf{ch}}_{2p-1}(\nabla^{V,u}, \nabla^V) \in A^{2p-1}(M \times X)$$

which satisfies

$$d\tilde{\mathbf{ch}}_{2p-1}(\nabla^{V,u}, \nabla^V) = \mathbf{ch}_{2p}(\nabla^{V,u}) - \mathbf{ch}_{2p}(\nabla^V) .$$

We now assume that X is smooth complex algebraic and that (V, g) is a geometric bundle (Definition 4.11). In particular, Assumption 4.9 is fulfilled. We claim that the Chern forms indeed belong to the subcomplex $A_{\log, Mf}(M \times X)$ with the correct weights.

First we assume that g globally comes from the compactification \bar{g} on $\bar{V} \rightarrow M \times \overline{Y \times X}$ via $\lambda : M \rightarrow Y$. Note that with Φ_λ as in (49)

$$\Phi_\lambda^* A(M \times \overline{Y \times X}) \subseteq \mathcal{W}_0 A_{\log, Mf}(M \times X) .$$

Since the Chern forms are natural and closed we have

$$\mathbf{ch}_{2p}(\nabla^V), \mathbf{ch}_{2p}(\nabla^{V,u}) \in \hat{\mathcal{W}}_{2p} A_{\log, Mf}^{2p}(M \times X) . \quad (51)$$

Similarly, if I is the unit interval with coordinate t we have

$$\mathbf{ch}_{2p}(t \mathbf{pr}_{M \times X}^* \nabla^V + (1-t) \mathbf{pr}_{M \times X}^* \nabla^{V,u}) \in \hat{\mathcal{W}}_{2p} A_{\log, Mf}^{2p}(I \times M \times X) .$$

Since by the proof of Lemma 4.1 integration along I preserves the subcomplex of logarithmic forms and the décalage $\hat{\mathcal{W}}$ of the weight filtration we conclude that

$$\tilde{\mathbf{ch}}_{2p-1}(\nabla^{\bar{V},u}, \nabla^{\bar{V}}) \in \hat{\mathcal{W}}_{2p} A_{\log, Mf}^{2p-1}(M \times X) . \quad (52)$$

In general, the geometry g extends to compactifications only locally on M . We now show that (51) and (52) still hold true. We write out the details for $\mathbf{ch}_{2p}(\nabla^V)$. The remaining cases are similar. Let $(U_\alpha)_{\alpha \in I}$ be an open covering of M such that $g|_{U_\alpha \times X}$ extends to a compactification, and let $(\chi_\alpha)_{\alpha \in I}$ be a partition of unity such that χ_α is supported in U_α . Then by (51) and the proof of Lemma 4.1 we have

$$\chi_\alpha \mathbf{ch}_{2p}(\nabla^V) \in \hat{\mathcal{W}}_{2p} A_{\log, Mf}^{2p}(M \times X) .$$

This implies

$$\mathbf{ch}_{2p}(\nabla^V) = \sum_{\alpha \in I} \chi_\alpha \mathbf{ch}_{2p}(\nabla^V) \in \hat{\mathcal{W}}_{2p} A_{\log, Mf}^{2p}(M \times X) ,$$

compare with the example (44).

Recall the Definition 4.3 of the cone $\mathbf{DR}_{Mf, \mathbb{C}}(p)$.

Definition 4.13. *We define the characteristic form of the geometric bundle (V, g^V) by*

$$\omega(p)(g^V) := \left(\mathbf{ch}_{2p}(\nabla^{V,u}) \oplus \mathbf{ch}_{2p}(\nabla^V), \tilde{\mathbf{ch}}_{2p-1}(\nabla^{V,u}, \nabla^V) \right) \in \mathbf{DR}_{Mf, \mathbb{C}}(p)(M \times X)^0 . \quad (53)$$

We further define

$$\omega(g^V) := \prod_{p \geq 0} \omega(p)(g^V) \in \mathbf{DR}_{Mf, \mathbb{C}}(M \times X)^0 . \quad (54)$$

The properties of the characteristic forms stated above show that $\omega(g^V)$ is well-defined and a cycle.

4.5 Bundles and algebraic K -theory

In the present section we define algebraic K -theory in terms of the symmetric monoidal stacks of bundles by Definition 2.1 and sheafification. We further introduce the stack of geometric bundles and verify that it can be used in the definition of K -theory as well.

Definition 4.14. *We let*

$$i\mathbf{Vect}_{Mf,\mathbb{C}} , \quad (\text{or } i\mathbf{Vect}_{Mf,\mathbb{Z}} \text{ respectively}) ,$$

be the symmetric monoidal stack on $\mathbf{Mf} \times \mathbf{Sm}_{Mf,\mathbb{C}}$ (or $\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$, respectively) which associates to $M \times X$ the symmetric monoidal (with respect to direct sum) groupoid of locally free, locally finitely generated sheaves of $\mathbf{pr}_X^ \mathcal{O}_X$ -modules on $M \times X$.*

We will call these sheaves bundles. We interpret

$$i\mathbf{Vect}_{Mf,\mathbb{C}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{Mf,\mathbb{C}}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]))$$

and

$$i\mathbf{Vect}_{Mf,\mathbb{Z}} \in \mathbf{Fun}^{desc}(\mathbf{S}_{Mf,\mathbb{Z}}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}])).$$

Note that these sheaves are locally constant in the manifold direction. This implies that $i\mathbf{Vect}_{Mf,\mathbb{C}}$ and $i\mathbf{Vect}_{Mf,\mathbb{Z}}$ are homotopy invariant in the sense of Definition A.2.

Recall that L denotes the sheafification functor (122).

Definition 4.15. *We define the homotopy invariant sheafs of algebraic K -theory spectra*

$$\mathbf{K}_{Mf,\mathbb{C}} \in \mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf,\mathbb{C}}, \mathbf{N}(\mathbf{Sp})[W^{-1}]) , \quad \mathbf{K}_{Mf,\mathbb{Z}} \in \mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf,\mathbb{Z}}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$$

by

$$\mathbf{K}_{Mf,\mathbb{C}} := L(K(i\mathbf{Vect}_{Mf,\mathbb{C}})), \quad \mathbf{K}_{Mf,\mathbb{Z}} := L(K(i\mathbf{Vect}_{Mf,\mathbb{Z}})) .$$

We have natural transformations

$$K(i\mathbf{Vect}_{Mf,\mathbb{C}}) \rightarrow \mathbf{K}_{Mf,\mathbb{C}} , \quad K(i\mathbf{Vect}_{Mf,\mathbb{Z}}) \rightarrow \mathbf{K}_{Mf,\mathbb{Z}} . \quad (55)$$

For X in $\mathbf{Sm}_{\mathbb{C}}$ or $\mathbf{Reg}_{\mathbb{Z}}$ the spectra $\mathbf{K}_{\mathbb{C}}(X)$ or $\mathbf{K}_{\mathbb{Z}}(X)$ represent generalized cohomology theories. By Lemma A.7 and the definition of $\mathbf{K}_{\mathbb{C}}$ respectively $\mathbf{K}_{\mathbb{Z}}$ in 3.5 we have equivalences

$$\mathbf{K}_{Mf,\mathbb{C}} \cong \mathbf{Sm}(\mathbf{K}_{\mathbb{C}}) , \quad \mathbf{K}_{Mf,\mathbb{Z}} \cong \mathbf{Sm}(\mathbf{K}_{\mathbb{Z}}) .$$

By (129) this immediately implies that for all $k \in \mathbb{Z}$ and manifolds M we have

$$\pi_k(\mathbf{K}_{Mf,\mathbb{C}}(M \times X)) \cong \mathbf{K}_{\mathbb{C}}(X)^{-k}(M)$$

or

$$\pi_k(\mathbf{K}_{Mf,\mathbb{Z}}(M \times X)) \cong \mathbf{K}_{\mathbb{Z}}(X)^{-k}(M) , \quad (56)$$

respectively.

Let $i\mathbf{Vect}_{Mf,\mathbb{C}}^{geom}$ be the symmetric monoidal stack on $\mathbf{Mf} \times \mathbf{Sm}_{\mathbb{C}}$ of geometric bundles as defined in 4.11. The stack of geometric bundles

$$i\mathbf{Vect}_{Mf,\mathbb{C}}^{geom} \in \mathbf{Fun}^{desc}(\mathbf{S}_{Mf,\mathbb{C}}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]))$$

has a forgetful transformation

$$i\mathbf{Vect}_{Mf,\mathbb{C}}^{geom} \rightarrow i\mathbf{Vect}_{Mf,\mathbb{C}} . \quad (57)$$

It is not homotopy invariant but we can enforce homotopy invariance by applying the functor \bar{s} defined in (124).

More precisely we have the following assertion.

Lemma 4.16. *The transformation (57) induces an equivalence*

$$\bar{\mathbf{s}}(\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{C}}^{geom})) \rightarrow \bar{\mathbf{s}}(\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{C}})) .$$

Proof. We use the notation \mathbf{s} defined in (123). A (p, q) -simplex v in the bisimplicial set $\mathbf{s}(\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{C}}))(M \times X)$ is given by a sequence of isomorphisms

$$V_0 \xrightarrow{\sim} \dots \xrightarrow{\sim} V_q$$

between bundles on $\Delta^p \times M \times X$.

One now checks that a (p, q) -simplex in $\mathbf{s}(\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{C}}^{geom}))(M \times X)$ over v is determined by a good geometry on V_q . By Lemma 4.12 $\mathbf{s}(\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{C}}^{geom}))(\bullet, q) \rightarrow \mathbf{s}(\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{C}}))(\bullet, q)$ is a Kan fibration with contractible fibres. Hence we get the required equivalence after applying the limit. \square

We now consider the arithmetic case. A geometry on a bundle $V \in i\mathbf{Vect}_{Mf,\mathbb{Z}}(M \times X)$ is by definition a good geometry on its base change $V \otimes \mathbb{C}$ to $M \times X \otimes \mathbb{C} = M \times (X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{C}))$ which in addition is invariant under the operation of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ in the second argument. In the following we make this precise.

Recall that the underlying complex manifold of $\overline{X \otimes \mathbb{C}}$ is the smooth manifold $X \otimes \mathbb{C}$ with its opposite holomorphic structure. For a vector bundle V on $M \times X$ we let $\overline{V \otimes \mathbb{C}}$ be the bundle $V \otimes \mathbb{C}$ equipped with the opposite complex structure.

Let $(\nabla^I, \bar{\partial})$ be the partial geometry (see Definition 4.7) on $V \otimes \mathbb{C}$. The same data can also be considered as a partial geometry on $\overline{V \otimes \mathbb{C}}$. Let $g := (\nabla^{II}, h^{V \otimes \mathbb{C}})$ extend the partial geometry to a geometry (Definition 4.8). Then we define the hermitean metric on $\overline{V \otimes \mathbb{C}}$ by

$$h^{\overline{V \otimes \mathbb{C}}}(\phi, \psi) := \overline{h^{V \otimes \mathbb{C}}(\phi, \psi)} .$$

The partial connection $\nabla^{II} = \bar{\partial} + \nabla^{1,0}$ can be considered as a partial connection on $\overline{V \otimes \mathbb{C}}$ which extends $\bar{\partial}$. We therefore can define the conjugated geometry by $\bar{g} := (\nabla^{II}, h^{\overline{V \otimes \mathbb{C}}})$. We now have a pair of canonical isomorphisms $u_{M \times X} : M \times X \otimes \mathbb{C} \rightarrow M \times \overline{X \otimes \mathbb{C}}$ and $U_{M \times X} : u_{M \times X}^* V \otimes \mathbb{C} \rightarrow \overline{V \otimes \mathbb{C}}$.

Definition 4.17. *Let V be a bundle over $M \times X$. We say that a geometry g on $V \otimes \mathbb{C}$ is $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ -invariant if*

$$(u_X, U_X)^* \bar{g} = g .$$

Definition 4.18. *A geometric bundle on $M \times X \in \mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ is a pair (V, g) of a bundle on $M \times X$ and a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ -invariant geometry g .*

We let $i\mathbf{Vect}_{Mf,\mathbb{Z}}$ be the symmetric monoidal stack of geometric bundles on $\mathbf{Mf} \times \mathbf{Reg}_{\mathbb{Z}}$ which we consider as an object

$$i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom} \in \mathbf{Fun}(\mathbf{S}_{Mf,\mathbb{Z}}, \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}])) .$$

We have a forgetful transformation

$$i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom} \rightarrow i\mathbf{Vect}_{Mf,\mathbb{Z}} \tag{58}$$

and the analogue of Lemma 4.16 with the same proof :

Lemma 4.19. *The transformation (58) induces an equivalence*

$$\bar{s}(\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom})) \rightarrow \bar{s}(\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{Z}})) .$$

4.6 The regulators $\mathbf{r}_{\mathbb{C}}^{\text{Beil}}$ and $\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}$

In this subsection we apply the machinery of Subsection 2.3 in order to construct the complex and arithmetic versions of the regulators $\mathbf{r}_{\mathbb{C}}^{\text{Beil}}$ and $\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}$.

We interpret the characteristic form ω defined in (54) as a characteristic cocycle

$$\omega : \pi_0(i\mathbf{Vect}_{Mf,\mathbb{C}}^{geom}) \rightarrow Z^0(\mathbf{DR}_{Mf,\mathbb{C}}) \quad (59)$$

in the sense of Definition 2.4. According to Definition 2.5 we get a regulator

$$K(\omega) : K(i\mathbf{Vect}_{Mf,\mathbb{C}}^{geom}) \rightarrow H(\mathbf{DR}_{Mf,\mathbb{C}}) .$$

Definition 4.20. *We define the complex version of the Beilinson regulator*

$$\mathbf{r}_{\mathbb{C}}^{\text{Beil}} : \mathbf{K}_{Mf,\mathbb{C}} \rightarrow H(\mathbf{DR}_{Mf,\mathbb{C}})$$

through the diagram

$$\begin{array}{ccccc} K(i\mathbf{Vect}_{Mf,\mathbb{C}}) & \xrightarrow[\text{Lemma A.3}]{\sim} & \bar{s}K(i\mathbf{Vect}_{Mf,\mathbb{C}}) & \xleftarrow[\text{Lemma 4.16}]{\sim} & \bar{s}K(i\mathbf{Vect}_{Mf,\mathbb{C}}^{geom}) & \xrightarrow{\bar{s}(K(\omega))} & \bar{s}H(\mathbf{DR}_{Mf,\mathbb{C}}) \\ \downarrow (55) & & & & & & \uparrow \sim \text{Lemma A.3} \\ \mathbf{K}_{Mf,\mathbb{C}} & \xrightarrow{\mathbf{r}_{\mathbb{C}}^{\text{Beil}}} & & & & & H(\mathbf{DR}_{Mf,\mathbb{C}}) . \end{array}$$

The factorization of the upper-right composition over the sheafification (the left vertical arrow) exists since $H(\mathbf{DR}_{Mf,\mathbb{C}})$ already satisfies descent.

We now consider the arithmetic situation.

Lemma 4.21. *If $(V, g) \in i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom}(M \times X)$ is a geometric bundle (Definition 4.18), then the characteristic form $\omega(g) \in Z^0(\mathbf{DR}_{Mf,\mathbb{C}}(M \times X \otimes \mathbb{C}))$ satisfies*

$$\overline{u_{M \times X}^* \omega(\bar{g})} = \omega(g) .$$

Proof. We have

$$u_{M \times X}^{-1,*} \omega(g) = \omega(u_{M \times X}^{-1,*} g) = \omega(\bar{g}) = \overline{\omega(g)} .$$

□

Using Lemma 4.21, we interpret the characteristic form ω as a characteristic cocycle

$$\omega : \pi_0(i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom}) \rightarrow Z^0(\mathbf{DR}_{Mf,\mathbb{Z}}) .$$

Definition 4.22. We define the arithmetic version of the Beilinson regulator

$$\mathbf{r}_{\mathbb{Z}}^{\text{Beil}} : \mathbf{K}_{Mf, \mathbb{Z}} \rightarrow H(\mathbf{DR}_{Mf, \mathbb{Z}})$$

through the diagram

$$\begin{array}{ccccc}
K(i\mathbf{Vect}_{Mf, \mathbb{Z}}) & \xrightarrow[\text{Lemma A.3}]{\sim} & \bar{s}K(i\mathbf{Vect}_{Mf, \mathbb{Z}}) & \xleftarrow[\text{Lemma 4.19}]{\sim} & \bar{s}K(i\mathbf{Vect}_{Mf, \mathbb{Z}}^{\text{geom}}) & \xrightarrow{\bar{s}(K(\omega))} & \bar{s}H(\mathbf{DR}_{Mf, \mathbb{Z}}) \\
\downarrow (55) & & & & & & \uparrow \sim \text{Lemma A.3} \\
\mathbf{K}_{Mf, \mathbb{Z}} & \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}} & & & & & H(\mathbf{DR}_{Mf, \mathbb{Z}}).
\end{array} \tag{60}$$

5 Differential algebraic K -theory

5.1 Definition of differential algebraic K -theory

A spectrum $E \in \mathbf{Sp}$ represents a cohomology theory E^* on the category of topological spaces. A differential extension \hat{E}^* combines the restriction of E^* to the category of smooth manifolds with information about characteristic forms. A standard choice for the differential form part is the de Rham complex ΩA with coefficients in the \mathbb{Z} -graded vector space $A := E^* \otimes \mathbb{R}$. In this case we have a canonical map

$$\text{rat} : \mathbf{Sm}(E) \rightarrow H(\Omega A) .$$

There is an immediate generalization of the theory to the case where A is a complex with possibly non-trivial differential such that $H^*(A) \cong E^*$. We refer to [BS10], [Bun12] for a detailed description of the axioms and more background on differential cohomology theories in general. Differential algebraic K -theory for number rings has been considered in detail in [BG], see also Subsection 5.5.1.

In the present paper we consider the differential extension of the algebraic K -theory of separated, regular schemes of finite type over \mathbb{Z} . Note that number rings correspond to schemes of relative dimension zero over \mathbb{Z} . Our choice of differential forms is the complex $\mathbf{DR}_{Mf, \mathbb{Z}}$. This complex is connected with algebraic K -theory by the regulator

$$\mathbf{K}_{Mf, \mathbb{Z}} \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}} H(\mathbf{DR}_{Mf, \mathbb{Z}})$$

given in (4.22). The set-up here generalizes the one explained in [BS10]. In the following we explain the differences. Usually, a differential extension of a cohomology theory is considered as a functor on the category of manifolds. Here it is a functor

$$M \times X \mapsto \hat{\mathbf{K}}(X)^0(M)$$

of two variables, a scheme X over $\mathbf{Spec}(\mathbb{Z})$ as above and the manifold M . Furthermore, the differential data is far from being strict in the sense explained in [Bun12]. Let us fix X and write

$$\mathbf{DR}_{\mathbb{Z}}(X)(M) := \mathbf{DR}_{Mf, \mathbb{Z}}(M \times X) . \tag{61}$$

Then $\mathbf{DR}_{\mathbb{Z}}(X)(M)$ is still different from the de Rham complex of M with coefficients in $\mathbf{K}_{-*}(X) \otimes \mathbb{R}$. Let us assume that X is proper in order to avoid the complications with compactifications. Then $\mathbf{DR}_{\mathbb{Z}}(X)(M)$ is a topological completion of the de Rham complex of M with coefficients in $\mathbf{DR}_{\mathbb{Z}}(X)(*)$. Even if X is zero-dimensional $\mathbf{DR}_{\mathbb{Z}}(X)(*)$ is a complex with non-trivial differential. Beilinson's conjectures described in Subsection 3.5 predict that, for proper X with potentially good reduction everywhere, the regulator induces an isomorphism of $\mathbf{K}_p(X) \otimes \mathbb{R} \rightarrow H^{-p}(\mathbf{DR}_{\mathbb{Z}}(X)(*))$ for $p \geq 2$, and its kernels and cokernels for other p are well understood and non-trivial in general. Hence the level of generality for differential extension $\widehat{\mathbf{K}}^0$ here includes that the characteristic map does not induce an equivalence $\mathbf{K}_{Mf,\mathbb{Z}} \wedge M\mathbb{R} \rightarrow H(\mathbf{DR}_{Mf,\mathbb{Z}})$, even if Beilinson's conjectures are assumed.

We turn now to the construction of the differential extension $\widehat{\mathbf{K}}^0$ which will be defined in terms of the differential algebraic K -theory spectrum. We denote the stupid truncation of a complex by $\sigma^{\geq 0}$.

Definition 5.1. *We define the differential algebraic K -theory spectrum*

$$\mathbf{Diff}(\mathbf{K}) \in \mathbf{Fun}(\mathbf{S}_{Mf,\mathbb{Z}}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$$

by the pull-back

$$\begin{array}{ccc} \mathbf{Diff}(\mathbf{K}) & \xrightarrow{R} & H(\sigma^{\geq 0} \mathbf{DR}_{Mf,\mathbb{Z}}) \\ \downarrow I & & \downarrow i \\ \mathbf{K}_{Mf,\mathbb{Z}} & \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}} & H(\mathbf{DR}_{Mf,\mathbb{Z}}) . \end{array} \quad (62)$$

Furthermore we define the differential algebraic K -theory (in degree zero) by

$$\widehat{\mathbf{K}}^0 := \pi_0(\mathbf{Diff}(\mathbf{K})) \in \mathbf{Fun}(\mathbf{S}_{Mf,\mathbb{Z}}, \mathbf{N}(\mathbf{Ab})) .$$

While $\mathbf{K}_{Mf,\mathbb{Z}}$ and $\mathbf{DR}_{Mf,\mathbb{Z}}$ are constant in the manifold direction (see Lemma 4.1 for the latter) the complex $\sigma^{\geq 0} \mathbf{DR}_{Mf,\mathbb{Z}}$ is no longer constant. Therefore $\mathbf{Diff}(\mathbf{K})$ is not constant as well. Hence the functor $\widehat{\mathbf{K}}^0$ is not homotopy invariant in the manifold direction. The deviation from homotopy invariance can be described by a homotopy formula which is formulated here in Lemma 5.6.

The maps R and I in the diagram (62) induce the structure maps (denoted by the same symbol)

$$R : \widehat{\mathbf{K}}^0 \rightarrow Z^0(\mathbf{DR}_{Mf,\mathbb{Z}}) , \quad I : \widehat{\mathbf{K}}^0 \rightarrow \pi_0(\mathbf{K}_{Mf,\mathbb{Z}}) .$$

The inclusion i of the non-negative part into the full Rham complex fits into a fibre sequence of complexes

$$\dots \rightarrow \sigma^{\leq -1} \mathbf{DR}_{Mf,\mathbb{Z}}[-1] \xrightarrow{a} \sigma^{\geq 0} \mathbf{DR}_{Mf,\mathbb{Z}} \xrightarrow{i} \mathbf{DR}_{Mf,\mathbb{Z}} \rightarrow \sigma^{\leq -1} \mathbf{DR}_{Mf,\mathbb{Z}} \rightarrow \dots .$$

The map marked by a in this sequence naturally induces the map

$$a : H^{-1}(\sigma^{\leq -1} \mathbf{DR}_{Mf,\mathbb{Z}}) \rightarrow \widehat{\mathbf{K}}^0 . \quad (63)$$

Since (62) is a pull-back square this map lives in an exact sequence

$$\pi_1(\mathbf{K}_{Mf,\mathbb{Z}}) \rightarrow H^{-1}(\sigma^{\leq -1} \mathbf{DR}_{Mf,\mathbb{Z}}) \xrightarrow{a} \widehat{\mathbf{K}}^0 \xrightarrow{I} \pi_0(\mathbf{K}_{Mf,\mathbb{Z}}) \rightarrow 0. \quad (64)$$

Definition 5.2. *We define the flat part of differential algebraic K-theory by*

$$\widehat{\mathbf{K}}_{flat}^0 := \ker \left(R : \widehat{\mathbf{K}}^0 \rightarrow Z^0(\mathbf{DR}_{Mf,\mathbb{Z}}) \right) .$$

5.2 The homotopy of $\text{Diff}(\mathbf{K})$

In this Subsection we calculate the homotopy groups of $\text{Diff}(\mathbf{K})$. We fix $X \in \mathbf{Reg}_{\mathbb{Z}}$ and write

$$\begin{aligned} \text{Diff}(\mathbf{K}(X))(M) &:= \text{Diff}(\mathbf{K})(X \times M) , \\ \widehat{\mathbf{K}}(X)^0(M) &:= \widehat{\mathbf{K}}^0(M \times X) , \end{aligned} \quad (65)$$

and

$$\widehat{\mathbf{K}}(X)_{flat}^0(M) := \widehat{\mathbf{K}}_{flat}^0(M \times X) . \quad (66)$$

Furthermore we let $\mathbf{K}_{\mathbb{Z}}(X)^*$, $\mathbf{K}_{\mathbb{Z}}(X)\mathbb{R}/\mathbb{Z}^*$ be the generalized cohomology theories represented by the spectra $\mathbf{K}_{\mathbb{Z}}(X)$ and its \mathbb{R}/\mathbb{Z} -version, respectively (compare with (56)). The calculation will partially depend on the validity of Beilinson's conjectures for X . We will indicate this dependence precisely at the corresponding places.

Proposition 5.3. *The homotopy groups of $\text{Diff}(\mathbf{K}(X))$ can be described as follows:*

1. *For $i \leq -1$ we have*

$$\pi_i(\text{Diff}(\mathbf{K}(X))) \cong K_{\mathbb{Z}}(X)^{-i} , \quad i \leq -1 .$$

2. *For $i \geq 1$ we have a natural map*

$$K_{\mathbb{Z}}(X)\mathbb{R}/\mathbb{Z}^{-i-1} \rightarrow \pi_i(\text{Diff}(\mathbf{K}(X))) . \quad (67)$$

It is an isomorphism if Beilinson's conjectures hold true for X .

3. *We have a sequence*

$$\begin{aligned} \mathbf{K}_{\mathbb{Z}}(X)\mathbb{R}/\mathbb{Z}^{-2} &\rightarrow \mathbf{K}_{\mathbb{Z}}(X)^{-1} \rightarrow H(\mathbf{DR}_{\mathbb{Z}}(X))^{-1} \xrightarrow{a} \widehat{\mathbf{K}}(X)^0 \xrightarrow{(I,R)} \\ &\rightarrow \mathbf{K}_{\mathbb{Z}}(X)^0 \times_{H(\mathbf{DR}_{\mathbb{Z}}(X))^0} Z^0(\mathbf{DR}_{\mathbb{Z}}(X)) \rightarrow 0 \end{aligned} \quad (68)$$

which is exact except possibly at $\mathbf{K}_{\mathbb{Z}}(X)^{-1}$. It is exact if Beilinson's conjectures hold true for X .

4. *We have an exact sequence*

$$\mathbf{K}_{\mathbb{Z}}(X)^{-1} \rightarrow H(\mathbf{DR}_{\mathbb{Z}}(X))^{-1} \xrightarrow{a} \widehat{\mathbf{K}}(X)_{flat}^0 \xrightarrow{I} \mathbf{K}_{\mathbb{Z}}(X)^0 \xrightarrow{r} H(\mathbf{DR}_{\mathbb{Z}}(X))^0 . \quad (69)$$

Proof. Since the inclusion map i in (62) induces an isomorphism on π_i for $i \leq -1$ and a surjection on π_0 we obtain from the long exact sequence associated to the pull-back that

$$\pi_i(\mathrm{Diff}(\mathbf{K}(X))) \cong \mathbf{K}_{\mathbb{Z}}(X)^{-i}, \quad i \leq -1.$$

We let $p_X : \mathbf{S}_{Mf} \rightarrow \mathbf{S}_{Mf, \mathbb{Z}}$ be given by

$$p_X(M) := M \times X. \quad (70)$$

The map

$$\mathbf{K}_{Mf, \mathbb{Z}} \mathbb{R} \rightarrow H(\mathbf{DR}_{Mf, \mathbb{Z}}) \quad (71)$$

induced by the regulator $\mathbf{r}_{\mathbb{Z}}^{\mathrm{Beil}}$ induces a map of pull-back diagrams from

$$\begin{array}{ccc} p_X^* \Sigma^{-1} \mathbf{K}_{Mf, \mathbb{Z}} \mathbb{R} / \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ p_X^* \mathbf{K}_{Mf, \mathbb{Z}} & \longrightarrow & p_X^* \mathbf{K}_{Mf, \mathbb{Z}} \mathbb{R} \end{array}$$

to

$$\begin{array}{ccc} \mathrm{Diff}(\mathbf{K}(X)) & \longrightarrow & p_X^* H(\sigma^{\geq 0} \mathbf{DR}_{Mf, \mathbb{Z}}(X)) \\ \downarrow & & \downarrow \\ p_X^* \mathbf{K}_{Mf, \mathbb{Z}} & \longrightarrow & p_X^* H(\mathbf{DR}_{Mf, \mathbb{Z}}(X)). \end{array} \quad (72)$$

The resulting map of left upper corners gives the map (67) in homotopy groups. For a manifold M we consider the morphism between the Atiyah-Hirzebruch spectral sequences for $\mathbf{K}_{\mathbb{Z}}(X) \mathbb{R}^*(M)$ and $H(\mathbf{DR}_{\mathbb{Z}}(X))^*(M)$ induced by (71). We assume that X satisfies Beilinson's conjecture 3.9. Then the map of E_2 -terms is an isomorphism on E_2^{pq} for all $q \leq -2$ and injective on $E_2^{0, -1}$. It follows that

$$\mathbf{K}_{\mathbb{Z}}(X) \mathbb{R}^{-i} \rightarrow H(\mathbf{DR}_{\mathbb{Z}}(X))^{-i}$$

is an isomorphism for $i \geq 2$ and injective for $i = 1$.

By an application of the Five Lemma we see that (67) is an isomorphism for $i \geq 1$.

The exactness of (68) is simply a part of the long exact sequence of homotopy groups associated with the pull-back diagram (72) where we have replaced $\pi_1(\mathrm{Diff}(\mathbf{K}(X)))$ at the left end by $\mathbf{K}_{\mathbb{Z}}(X) \mathbb{R} / \mathbb{Z}^{-2}$ using (67). Hence it is exact except possibly at $\mathbf{K}_{\mathbb{Z}}(X)^{-1}$. By the above, it is exact everywhere if we assume Beilinson's conjecture for X .

The exactness of (69) follows from the unconditional part of exactness of (68) by pull-back along the injective transformation

$$\ker \left(\mathbf{K}_{\mathbb{Z}}(X)^0 \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\mathrm{Beil}}} H(\mathbf{DR}_{\mathbb{Z}}(X))^0 \right) \xrightarrow{x \mapsto (x, 0)} \mathbf{K}(\mathbb{Z}(X))^0 \times_{H(\mathbf{DR}_{\mathbb{Z}}(X))^0} Z^0(\mathbf{DR}_{\mathbb{Z}}(X)).$$

□

5.3 The geometric cycle map

Let M be a smooth manifold and X a scheme in $\mathbf{Reg}_{\mathbb{Z}}$. A bundle $V \in i\mathbf{Vect}_{Mf,\mathbb{Z}}(M \times X)$ (cf. Definition 4.14) gives rise to a class

$$\mathrm{cycl}(V) \in \pi_0(\mathbf{K}_{Mf,\mathbb{Z}}(M \times X)) \cong \mathbf{K}_{\mathbb{Z}}(X)^0(M) .$$

The natural transformation cycl is called the topological cycle map, see (74) for details. If g^V is a good geometry on V so that (V, g^V) is a geometric bundle (cf. Definition 4.18), then we have the characteristic form $\omega(g^V) \in Z^0(\mathbf{DR}_{Mf,\mathbb{Z}}(M \times X))$ defined in (53) which represents the class of the regulator

$$\mathbf{r}_{\mathbb{Z}}^{\mathrm{Beil}}(\mathrm{cycl}(V)) \in H^0(\mathbf{DR}_{Mf,\mathbb{Z}}(M \times X)) \cong H(\mathbf{DR}_{\mathbb{Z}}(X))^0(M)$$

The main result of the present Subsection, Definition 5.5, is the construction of a geometric cycle map $\widehat{\mathrm{cycl}}$ which maps (V, g^V) to the differential algebraic K -theory class

$$\widehat{\mathrm{cycl}}(V, g^V) \in \widehat{\mathbf{K}}(X)^0(M)$$

such that

$$R(\widehat{\mathrm{cycl}}(V, g^V)) = \omega(g^V) , \quad I(\widehat{\mathrm{cycl}}(V, g^V)) = \mathrm{cycl}(V) . \quad (73)$$

A first version of topological and geometric cycle maps for number rings has been introduced in [BG, Sec. 3.6]. The approach presented in the present paper is simpler, more functorial, and works for higher-dimensional X as well.

We start with a precise description of the topological cycle map. We consider the functor

$$\pi_0(i\mathbf{Vect}_{Mf,\mathbb{Z}}) : \mathbf{Mf}^{op} \times \mathbf{Reg}_{\mathbb{Z}}^{op} \rightarrow \mathbf{Mon}(\mathbf{Set}) ,$$

see Subsection 4.5.

Definition 5.4. *The natural map of monoids*

$$\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{Z}}) \rightarrow \Omega B(\mathbf{N}(i\mathbf{Vect}_{Mf,\mathbb{Z}})) \cong \Omega^\infty K(i\mathbf{Vect}_{Mf,\mathbb{Z}}) \rightarrow \Omega^\infty \mathbf{K}_{Mf,\mathbb{Z}}$$

induces the topological cycle map

$$\mathrm{cycl} : \pi_0(i\mathbf{Vect}_{Mf,\mathbb{Z}}) \rightarrow \pi_0(\Omega^\infty \mathbf{K}_{Mf,\mathbb{Z}}) \cong \pi_0(\mathbf{K}_{Mf,\mathbb{Z}}) \quad (74)$$

as a map in $\mathbf{Fun}(\mathbf{S}_{Mf,\mathbb{Z}}, \mathbf{N}(\mathbf{Mon}(\mathbf{Set})))$.

In order to define the geometric cycle map we extend the diagram (60) used in the construction of the regulator, Definition 4.22, to the commuting diagram

$$\begin{array}{ccccccc} K(i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom}) & \xrightarrow[\text{(9)}]{K(\omega)} & K(Z^0(\mathbf{DR}_{Mf,\mathbb{Z}})) & \xrightarrow[\text{(10)}]{\sim} & H(Z^0(\mathbf{DR}_{Mf,\mathbb{Z}})) & \longrightarrow & H(\sigma^{\geq 0} \mathbf{DR}_{Mf,\mathbb{Z}}) \\ \downarrow & & \searrow & & & & \downarrow \\ K(i\mathbf{Vect}_{Mf,\mathbb{Z}}) & \longrightarrow & \bar{s}K(i\mathbf{Vect}_{Mf,\mathbb{Z}}) & \xleftarrow{\sim} & \bar{s}K(i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom}) & \xrightarrow{\bar{s}K(\omega)} & \bar{s}H(\mathbf{DR}_{Mf,\mathbb{Z}}) \\ \downarrow & & & & & & \uparrow \sim \\ \mathbf{K}_{Mf,\mathbb{Z}} & \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\mathrm{Beil}}} & & & & & H(\mathbf{DR}_{Mf,\mathbb{Z}}) . \end{array}$$

By the universal property of the pull-back defining $\mathbf{Diff}(\mathbf{K})$ the outer square defines a canonical map

$$K(i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom}) \rightarrow \mathbf{Diff}(\mathbf{K}) \quad (75)$$

of diagrams of spectra.

Definition 5.5. *We define the geometric cycle map*

$$\widehat{\mathbf{cyc1}} : \pi_0(i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom}) \rightarrow \widehat{\mathbf{K}}^0$$

as the transformation in $\mathbf{Fun}(\mathbf{S}_{Mf,\mathbb{Z}}, \mathbf{N}(\mathbf{Mon}(\mathbf{Set})))$ given by composition

$$\pi_0(i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom}) \rightarrow \pi_0(K(i\mathbf{Vect}_{Mf,\mathbb{Z}}^{geom})) \xrightarrow{(75)} \pi_0(\mathbf{Diff}(\mathbf{K})) .$$

It is clear from the construction that this geometric cycle map satisfies the conditions (73).

5.4 Homotopy formulas

5.4.1 The manifold direction

Let $I := [0, 1]$ be the unit interval, $M \in \mathbf{Mf}$ be a smooth manifold, and $X \in \mathbf{Sm}_{\mathbb{C}}$ be a smooth algebraic variety. As observed in the proof of Lemma 4.1 the integral

$$\int_I : A^*(I \times M \times X) \rightarrow A^{*-1}(M \times X)$$

induces a morphism

$$\int_I : \mathbf{DR}_{Mf,\mathbb{C}}(I \times M \times X) \rightarrow \mathbf{DR}_{Mf,\mathbb{C}}(M \times X)[-1] .$$

Explicitly, if $(\omega_{\mathbb{R}} \oplus \omega, \tilde{\omega})$ is an element in the cone then

$$\int_I (\omega_{\mathbb{R}} \oplus \omega, \tilde{\omega}) = \left(\int_I \omega_{\mathbb{R}} \oplus \int_I \omega, - \int_I \tilde{\omega} \right) .$$

Let $i_0, i_1 : M \rightarrow I \times M$ be the inclusions corresponding to the end points of the interval. By Stoke's theorem, for $\omega \in \mathbf{DR}_{Mf,\mathbb{C}}(I \times M \times X)$ we then have the identity

$$\int_I d\omega = i_1^* \omega - i_0^* \omega - d \int_I \omega \quad (76)$$

in $\mathbf{DR}_{Mf,\mathbb{C}}(M \times X)$. A similar formula holds true for $X \in \mathbf{Reg}_{\mathbb{Z}}$ and $\omega \in \mathbf{DR}_{Mf,\mathbb{Z}}$ (compare Definition 4.4).

Differential algebraic K -theory $\widehat{\mathbf{K}}^0$ admits a homotopy formula in the manifold direction. Assume that $M \in \mathbf{Mf}$ is a smooth manifold and X is a scheme in $\mathbf{Reg}_{\mathbb{Z}}$. Recall that we denote by a the natural map $\mathbf{DR}_{Mf,\mathbb{Z}}^{-1}(M \times X)/\mathbf{im}(d) \xrightarrow{(63)} \widehat{\mathbf{K}}^0(M \times X)$.

Lemma 5.6. *If $x \in \widehat{\mathbf{K}}^0(I \times M \times X)$, then we have*

$$i_1^*(x) - i_0^*(x) = a\left(\int_I R(x)\right).$$

Proof. The proof is a formal consequence of (76) and the exactness of (68) at $\widehat{\mathbf{K}}^0$. The argument is parallel to the proof of [BS10, Lemma 5.1] or Proposition 5.8 below. \square

As an application we discuss how the geometric cycle class changes when one varies the geometry. Thus we let V be a locally free sheaf of $\mathbf{pr}_X^* \mathcal{O}_X$ -modules on $M \times X$ and g_0^V, g_1^V two geometries. We know from Lemma 4.12 that there exists a geometry \hat{g}^V on the pull-back $\mathbf{pr}_{M \times X}^* V$ on $I \times M \times X$ such that $i_j^*(\hat{g}^V) = g_j^V, j = 0, 1$. Hence the homotopy formula and (73) yield

$$\widehat{\text{cyc1}}(V, g_1^V) - \widehat{\text{cyc1}}(V, g_0^V) = a\left(\int_I \omega(\hat{g}^V)\right). \quad (77)$$

We consider the following special case. We fix the partial connection ∇^{II} and assume moreover that the total connection ∇ is flat. We consider a metric $h_0 = h$ and a rescaled metric $h_1 = \lambda h$ for some $\lambda \in (0, \infty) \subset \mathbb{R}$.

Lemma 5.7.

$$\widehat{\text{cyc1}}(V, (\nabla^{II}, \lambda h)) - \widehat{\text{cyc1}}(V, (\nabla^{II}, h)) = a\left((0, 0, -\frac{1}{2} \dim(V) \log(\lambda))\right). \quad (78)$$

where $(0, 0, -\frac{1}{2} \dim(V) \log(\lambda)) \in \mathbf{DR}_{Mf, \mathbb{C}}^{-1}(1)(M \times X)$.

Proof. We choose a smooth function ρ on I with $\rho(0) = 1, \rho(1) = \lambda$. Then $\hat{g}^V = (\mathbf{pr}_{M \times X}^* \nabla^{II}, \rho \cdot \mathbf{pr}_{M \times X}^* h)$ is a good geometry on $\mathbf{pr}_{M \times X}^* V$. We can compute the integral in (77) explicitly. Denoting the total connection on $\mathbf{pr}_{M \times X}^*(V)$ by $\hat{\nabla}$ we find that its adjoint is simply given by

$$\hat{\nabla}^* = \mathbf{pr}_{M \times X}^*(\nabla^{*h}) + d \log(\rho).$$

The curvature of $\hat{\nabla}^u$ is the pull-back of the curvature of ∇^{u_h} , hence

$$\int_I \mathbf{ch}_{2p}(\hat{\nabla}^u) = 0.$$

Write $\omega := \nabla^{u_h} - \nabla$, $\hat{\omega} := \hat{\nabla}^u - \hat{\nabla} = \mathbf{pr}_{M \times X}^* \omega + \frac{1}{2} d \log(\rho)$. Since $\hat{\nabla}$ is flat the transgression $\tilde{\mathbf{ch}}_{2p-1}(\hat{\nabla}^u, \hat{\nabla})$ depends on the connections via the form $\text{Tr}(\hat{\omega}^{2p-1})$. We have

$$\hat{\omega}^{2p-1} = \mathbf{pr}_{M \times X}^* \omega^{2p-1} + \frac{p}{2} d \log(\rho) \wedge \mathbf{pr}_{M \times X}^* \omega^{2p-2}.$$

Since $\text{Tr}(\omega^{2p-2}) = 0$ for $p > 1$ we get $\tilde{\mathbf{ch}}_{2p-1}(\hat{\nabla}^u, \hat{\nabla}) = \mathbf{pr}_{M \times X}^* \tilde{\mathbf{ch}}_{2p-1}(\nabla^u, \nabla)$ for $p > 1$. Furthermore, $\tilde{\mathbf{ch}}_1(\hat{\nabla}^u, \hat{\nabla}) = \mathbf{pr}_{M \times X}^* \tilde{\mathbf{ch}}_1(\nabla^u, \nabla) + \frac{\dim(V)}{2} d \log(\rho)$. We finally get

$$\int_I \tilde{\mathbf{ch}}_{2p-1}(\hat{\nabla}^u, \hat{\nabla}) = \begin{cases} \frac{1}{2} \log(\lambda) \dim V & \text{if } p = 1 \\ 0 & \text{else.} \end{cases}$$

This implies (78). □

5.4.2 The algebraic direction

It is known that absolute Hodge cohomology is homotopy invariant in the algebraic direction in the sense that the natural projection induces a quasi-isomorphism

$$\mathbf{DR}_{Mf,\mathbb{C}}(M \times X) \xrightarrow[\sim]{\mathbf{pr}_{M \times X}^*} \mathbf{DR}_{Mf,\mathbb{C}}(M \times X \times \mathbb{A}_{\mathbb{C}}^1).$$

In particular, the two inclusions $u_0, u_1 : M \times X \hookrightarrow M \times X \times \mathbb{A}_{\mathbb{C}}^1$ given by the points $0, 1 \in \mathbb{A}_{\mathbb{C}}^1(\mathbb{C})$ induce the same map on cohomology. We expect that there exists an integration operator \int_H in the algebraic direction which satisfies a formula similar to (76). However, the actual construction of such an operator seems to be quite complicated, and we content ourselves with the analogous formula for $\mathbb{P}_{\mathbb{C}}^1$. This is enough for our purposes in the current paper.

On the space of forms $A(M \times X \times \mathbb{CP}^1)$ we consider the operator Q given by

$$Q(\omega) = \mathbf{pr}_{M \times X}^* u_0^* \omega + \mathbf{pr}_{M \times X}^* \left(\int_{M \times X \times \mathbb{CP}^1 / M \times X} \omega \right) \wedge \text{vol}_{\mathbb{CP}^1}$$

with the normalized volume form $\text{vol}_{\mathbb{CP}^1}$. It preserves the subcomplexes $\hat{\mathcal{W}}_k A_{\log, Mf, \mathbb{R}}$ and $\mathcal{F}^p A_{\log, Mf}$ and hence induces an operator Q on $\mathbf{DR}_{Mf,\mathbb{C}}(M \times X \times \mathbb{CP}^1)$. Below, we will construct a homotopy

$$h : \mathbf{DR}_{Mf,\mathbb{C}}(M \times X \times \mathbb{CP}^1) \rightarrow \mathbf{DR}_{Mf,\mathbb{C}}(M \times X \times \mathbb{CP}^1)[-1] \quad (79)$$

which is natural in $M \times X$ and such that

$$dh + hd = \text{id} - Q. \quad (80)$$

Assuming for the moment that we have already constructed such a homotopy h we let $u_t : M \times X \hookrightarrow M \times X \times \mathbb{CP}^1$ be the inclusion given by $t \in \mathbb{C} \subset \mathbb{CP}^1$.

We consider the flow $\Phi_t : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ given by $\Phi_t(z) = z + t$. We extend this flow constantly along the $M \times X$ -direction. Then we have $u_t^* = u_0^* \Phi_t^*$. It follows that

$$u_1^* \omega - u_0^* \omega = \int_0^1 \frac{d}{dt} u_0^* \Phi_t^* \omega \, dt.$$

We now insert (80) and observe that $u_0^* \Phi_t^* Q(\omega)$ does not depend on t . We get

$$u_1^* \omega - u_0^* \omega = \int_0^1 \frac{d}{dt} u_0^* \Phi_t^* (hd + dh) \omega \, dt.$$

We rewrite this as

$$u_1^* \omega - u_0^* \omega = \int_0^1 u_0^* \frac{d}{dt} \Phi_t^* h(d\omega) \, dt + d \int_0^1 u_0^* \frac{d}{dt} \Phi_t^* h(\omega) \, dt.$$

Hence we are led to define

$$\int_{II} \omega := \int_0^1 u_0^* \frac{d}{dt} \Phi_t^* h(\omega) dt . \quad (81)$$

The above formula then reads

$$u_1^* \omega - u_0^* \omega = \int_{II} d\omega + d \int_{II} \omega . \quad (82)$$

Again, we have a similar formula for $X \in \mathbf{Reg}_{\mathbb{Z}}$ and $\omega \in \mathbf{DR}_{Mf, \mathbb{Z}}(M \times X \times \mathbb{P}_{\mathbb{Z}}^1)$. It now remains to find the homotopy h (79). Assume that we have natural transformations

$$h_{\mathbb{R}}, h_{\mathcal{F}}, \tilde{h} : \hat{\mathcal{W}}_p A_{\log}(M \times X \times \mathbb{CP}^1) \rightarrow \hat{\mathcal{W}}_p A_{\log}(M \times X \times \mathbb{CP}^1)[-1]$$

and

$$r_{\mathbb{R}}, r_{\mathcal{F}} : \hat{\mathcal{W}}_p A_{\log}(M \times X \times \mathbb{CP}^1) \rightarrow \hat{\mathcal{W}}_p A_{\log}(M \times X \times \mathbb{CP}^1)[-2]$$

such that

$$\begin{aligned} dh_{\mathbb{R}} + h_{\mathbb{R}} d &= \text{id} - Q = dh_{\mathcal{F}} + h_{\mathcal{F}} d , & d\tilde{h} + \tilde{h} d &= \text{id} - Q \\ dr_{\mathbb{R}} - r_{\mathbb{R}} d &= h_{\mathbb{R}} - \tilde{h} , & dr_{\mathcal{F}} - r_{\mathcal{F}} d &= h_{\mathcal{F}} - \tilde{h} , \end{aligned}$$

and $h_{\mathbb{R}}$ preserves $A_{\log, \mathbb{R}}$ and $h_{\mathcal{F}}$ preserves $\mathcal{F}^p A_{\log}$. Then we define

$$h = \prod_{p \geq 0} h_{\mathbf{DR}(p)} : \mathbf{DR}_{Mf, \mathbb{C}}(M \times X \times \mathbb{CP}^1) \rightarrow \mathbf{DR}_{Mf, \mathbb{C}}(M \times X \times \mathbb{CP}^1)[-1]$$

by

$$h_{\mathbf{DR}(p)}(\omega_{\mathbb{R}}, \omega_{\mathcal{F}}, \tilde{\omega}) := (h_{\mathbb{R}} \omega, h_{\mathcal{F}} \omega, -\tilde{h} \tilde{\omega} + r_{\mathbb{R}} \omega_{\mathbb{R}} - r_{\mathcal{F}} \omega_{\mathcal{F}}) .$$

We check:

$$dh_{\mathbf{DR}(p)}(\omega_{\mathbb{R}}, \omega_{\mathcal{F}}, \tilde{\omega}) = (dh_{\mathbb{R}} \omega_{\mathbb{R}}, dh_{\mathcal{F}} \omega_{\mathcal{F}}, d\tilde{h} \tilde{\omega} - dr_{\mathbb{R}} \omega_{\mathbb{R}} + dr_{\mathcal{F}} \omega_{\mathcal{F}} + h_{\mathbb{R}} \omega_{\mathbb{R}} - h_{\mathcal{F}} \omega_{\mathcal{F}}) .$$

$$\begin{aligned} h_{\mathbf{DR}(p)} d(\omega_{\mathbb{R}}, \omega_{\mathcal{F}}, \tilde{\omega}) &= h_{\mathbf{DR}(p)}(d\omega_{\mathbb{R}}, d\omega_{\mathcal{F}}, -d\tilde{\omega} + \omega_{\mathbb{R}} - \omega_{\mathcal{F}}) \\ &= (h_{\mathbb{R}} d\omega_{\mathbb{R}}, h_{\mathcal{F}} d\omega_{\mathcal{F}}, \tilde{h} d\tilde{\omega} - \tilde{h} \omega_{\mathbb{R}} + \tilde{h} \omega_{\mathcal{F}} + r_{\mathbb{R}} d\omega_{\mathbb{R}} - r_{\mathcal{F}} d\omega_{\mathcal{F}}) . \end{aligned}$$

We get

$$(dh_{\mathbf{DR}(p)} + h_{\mathbf{DR}(p)} d)(\omega_{\mathbb{R}}, \omega_{\mathcal{F}}, \tilde{\omega})_{\mathbb{R}} = dh_{\mathbb{R}} \omega_{\mathbb{R}} + h_{\mathbb{R}} d\omega_{\mathbb{R}} = \omega_{\mathbb{R}} - Q(\omega_{\mathbb{R}}) .$$

$$(dh_{\mathbf{DR}(p)} + h_{\mathbf{DR}(p)} d)(\omega_{\mathbb{R}}, \omega_{\mathcal{F}}, \tilde{\omega})_{\mathcal{F}} = dh_{\mathcal{F}} \omega_{\mathcal{F}} + h_{\mathcal{F}} d\omega_{\mathcal{F}} = \omega_{\mathcal{F}} - Q(\omega_{\mathcal{F}}) ,$$

and

$$\begin{aligned} (dh_{\mathbf{DR}(p)} + h_{\mathbf{DR}(p)} d)(\omega_{\mathbb{R}}, \omega_{\mathcal{F}}, \tilde{\omega})_{\sim} &= d\tilde{h} \tilde{\omega} - dr_{\mathbb{R}} \omega_{\mathbb{R}} + dr_{\mathcal{F}} \omega_{\mathcal{F}} + h_{\mathbb{R}} \omega_{\mathbb{R}} - h_{\mathcal{F}} \omega_{\mathcal{F}} \\ &\quad + \tilde{h} d\tilde{\omega} - \tilde{h} \omega_{\mathbb{R}} + \tilde{h} \omega_{\mathcal{F}} + r_{\mathbb{R}} d\omega_{\mathbb{R}} - r_{\mathcal{F}} d\omega_{\mathcal{F}} \\ &= \tilde{\omega} - Q(\tilde{\omega}) \end{aligned}$$

In the following we show how the Kähler package induces appropriate homotopies. Let Y be a compact Kähler manifold. Then we consider the operator $G : A(Y) \rightarrow A(Y)$ which vanishes on the harmonic forms $\mathcal{H} := \ker(\Delta)$ and is Δ^{-1} on the \mathcal{H}^\perp . We define

$$h_{\mathbb{R}} := d^*G, \quad h_{\mathcal{F}} := 2\bar{\partial}^*G, \quad \tilde{h} := d^*G. \quad (83)$$

Using the adjoint L^* of $L : \alpha \mapsto \omega \wedge \alpha$ (with ω the Kähler form) we define

$$r_{\mathcal{F}} := iL^*G, \quad r_{\mathbb{R}} := 0. \quad (84)$$

We use that

$$\Delta = dd^* + d^*d = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}).$$

We have

$$dh_{\mathbb{R}} + h_{\mathbb{R}}d = \text{id} - \text{pr}_{\mathcal{H}}, \quad dh_{\mathcal{F}} + h_{\mathcal{F}}d = \text{id} - \text{pr}_{\mathcal{H}}.$$

Finally using

$$\tilde{h} - h_{\mathcal{F}} = (\partial^* - \bar{\partial}^*)G, \quad [d, L^*] = i\partial^* - i\bar{\partial}^*$$

we get

$$[d, r_{\mathcal{F}}] = h_{\mathcal{F}} - \tilde{h}.$$

We apply this in the case $Y := \mathbb{CP}^1$. In general, a form in $A(M \times X \times \mathbb{CP}^1)$ can be considered as a form on $M \times X$ with values in $A(\mathbb{CP}^1)$. In this way the homotopy operators just introduced for \mathbb{CP}^1 induce the desired homotopy operators for $M \times X \times \mathbb{CP}^1$. One checks that these have the necessary properties.

We retain the notation from the beginning of this Subsection with $M \in \mathbf{Mf}$ and $X \in \mathbf{Reg}_{\mathbb{Z}}$ and let $u_0, u_1 : M \times X \rightarrow M \times X \times \mathbb{P}_{\mathbb{Z}}^1$ denote the two inclusions given by the points $0, 1 \in \mathbb{P}_{\mathbb{Z}}^1(\mathbb{Z})$.

Proposition 5.8. *We consider $x \in \widehat{\mathbf{K}}^0(M \times X \times \mathbb{P}_{\mathbb{Z}}^1)$. We assume that there exists a class $y \in \widehat{\mathbf{K}}^0(M \times X \times \mathbb{P}_{\mathbb{Z}}^1)$ such that $I(y) = I(x)$ and $\int_{II} R(y) = 0$. Then*

$$(u_1^* - u_0^*)x - (u_1^* - u_0^*)y = a\left(\int_{II} R(x)\right).$$

Proof. By the exactness of (64) at $\widehat{\mathbf{K}}^0$ we can find an element $\omega \in \mathbf{DR}_{Mf, \mathbb{Z}}^{-1}(M \times X \times \mathbb{P}_{\mathbb{Z}}^1)$ such that $x - y = a(\omega)$. It follows that

$$(u_1^* - u_0^*)x - (u_1^* - u_0^*)y = a(u_1^*\omega - u_0^*\omega) \stackrel{(82)}{=} a\left(\int_{II} d\omega\right) = a\left(\int_{II} R(x)\right)$$

since $R(y)$ is annihilated by \int_{II} . □

Remark 5.9. *It is likely that such a class y always exists. E.g. this is the case if $I(x)$ is the pull-back of a class $\bar{y} \in \mathbf{K}_{Mf, \mathbb{Z}}^0(M \times X)$. We expect that, if we had a multiplicative version of differential algebraic K -theory, the existence of y follows from the projective bundle formula in algebraic K -theory. On the other hand, if x is given as the geometric cycle class of some geometric bundle one can often construct a suitable y by geometric means.*

5.5 Number rings

5.5.1 A review of [BG]

In this Subsection we review the construction of differential algebraic K -theory \widehat{KR}^0 as introduced in [BG, Sec. 2.2]. Let R be a number ring, i.e. the ring of integers of a finite field extension $\mathbb{Q} \subseteq k$. We consider the scheme $X = \mathbf{Spec}(R) \in \mathbf{Reg}_{\mathbb{Z}}$, which is of relative dimension 0 over $\mathbf{Spec}(\mathbb{Z})$, and its algebraic K -theory spectrum $KR := \mathbf{K}_{\mathbb{Z}}(X) \in \mathbf{N}(\mathbf{Sp}^{\geq 0})[W^{-1}]$. We define the \mathbb{Z} -graded vector space $A^* := \pi_{-*}(KR) \otimes \mathbb{R}$ which we consider as a chain complex with trivial differential. We let

$$\Omega A \in \mathbf{Sh}_{\mathbf{Ch}}(\mathbf{Mf}) , \quad M \mapsto A_{\mathbb{R}}(M) \otimes A$$

be the sheaf of chain complexes of real differential forms with coefficients in A .

We abbreviate the smash product with the Moore spectrum $M\mathbb{R}$ of the abelian group \mathbb{R} by $KR\mathbb{R} := KR \wedge M\mathbb{R}$. Then we have a canonical map $c : KR \rightarrow KR\mathbb{R} \xrightarrow{\sim} H(A)$, where $H(A) \in \mathbf{N}(\mathbf{Sp})[W^{-1}]$ is the Eilenberg-MacLane spectrum of A (see A.5). We furthermore have a de Rham equivalence $j : H(\Omega A) \xrightarrow{\sim} \mathbf{Sm}(H(A))$ (see Lemma 4.5) where $\mathbf{Sm}(E) \in \mathbf{Fun}^{desc}(\mathbf{S}_{Mf}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$ denotes the smooth function spectrum of a spectrum $E \in \mathbf{N}(\mathbf{Sp})[W^{-1}]$ (see A.10). We use the canonical map c and the inverse of the de Rham equivalence in order to define the map

$$\mathbf{rat} : \mathbf{Sm}(KR) \xrightarrow{c} \mathbf{Sm}(H(A)) \xleftarrow[\sim]{j} H(\Omega A) . \quad (85)$$

The differential function spectrum

$$\mathbf{Diff}(KR) \in \mathbf{Fun}^{desc}(\mathbf{S}_{Mf}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$$

is given as the pull-back

$$\begin{array}{ccc} \mathbf{Diff}(KR) & \xrightarrow{R} & H(\sigma^{\geq 0}\Omega A) \\ \downarrow I & & \downarrow \\ \mathbf{Sm}(KR) & \xrightarrow{\mathbf{rat}} & H(\Omega A) \end{array} \quad (86)$$

in $\mathbf{Fun}^{desc}(\mathbf{S}_{Mf}, \mathbf{N}(\mathbf{Sp})[W^{-1}])$. The version of differential algebraic K -theory in [BG] is the functor

$$\widehat{KR}^0 := \pi_0(\mathbf{Diff}(KR)) \in \mathbf{Ab}^{\mathbf{Mf}^{op}} . \quad (87)$$

The maps denoted by R and I in the diagram (86) induce maps

$$R : \widehat{KR}^0 \rightarrow Z^0(\Omega A) , \quad I : \widehat{KR}^0 \rightarrow KR^0 ,$$

and we have a map

$$a : \Omega A^{-1}/\mathbf{im}(d) \rightarrow \widehat{KR}^0 .$$

We shall use the exact sequence ([BG, Def. 2.2 (iii)], the analog of (64))

$$KR^{-1} \rightarrow \Omega A^{-1}/\mathbf{im}(d) \xrightarrow{a} \widehat{KR}^0 \xrightarrow{I} KR^0 \rightarrow 0 . \quad (88)$$

5.5.2 Comparison of two versions of differential algebraic K -theory

We want to give the precise relation between the differential algebraic K -theory \widehat{KR}^0 introduced in [BG] (or (87)) and $\widehat{\mathbf{K}}(X)^0$ in the notation (65). We further use the notation $p_X^* \mathbf{DR}_{Mf, \mathbb{Z}} \in \mathbf{PSh}_{\mathbf{Ch}}(\mathbf{Mf})$ with p_X as in (70).

We must relate the two versions of forms ΩA and $p_X^* \mathbf{DR}_{Mf, \mathbb{Z}}$ corresponding to \widehat{KR}^0 and $\widehat{\mathbf{K}}(X)^0$. We write elements $\omega \in \mathbf{DR}_{Mf, \mathbb{Z}}(M \times X)$ in the form

$$\omega = (\omega_{\mathbb{R}}(p) \oplus \omega(p), \tilde{\omega}(p))_{p \geq 0} \in \prod_{p \geq 0} \mathbf{DR}_{Mf, \mathbb{Z}}(p)(M \times X) . \quad (89)$$

Here $\omega_{\mathbb{R}}(p) \in (2\pi i)^p A_{\mathbb{R}}(M \times X)[2p]$, $\omega(p) \in \mathcal{F}^p A(M \times X)[2p]$, and $\tilde{\omega}(p) \in A(M \times X)[2p-1]$, where we use the notation $A(M \times X) := [A(M \times X(\mathbb{C}))]^{\text{Gal}(\mathbb{C}/\mathbb{R})}$ where $\text{Gal}(\mathbb{C}/\mathbb{R})$ acts on the set $X(\mathbb{C})$ of complex points of X and the differential forms by complex conjugation. Furthermore, $\mathcal{F}^p A(M \times X) = 0$ for $p \geq 1$ and $\mathcal{F}^0 A(M \times X) = A(M \times X)$. We define

$$\text{Re}(\tilde{\omega}(p)) := (2\pi i)^p \Re((2\pi i)^{-p} \tilde{\omega}(p)) , \quad \text{Im}(\tilde{\omega}(p)) := (2\pi i)^{p+1} \Re((2\pi i)^{-p-1} \tilde{\omega}(p)) , \quad (90)$$

where \Re denotes the usual real part. Note that for every complex point $\sigma \in X(\mathbb{C})$ we have an evaluation

$$\text{ev}_{\sigma} : A(M \times X) \rightarrow A(M) .$$

We will use the following results about Beilinson's regulator for R : Combining Borel's results [Bor74] concerning the ranks of the $K_i(R)$, $i \geq 0$, and the comparison between the regulators of Borel and Beilinson [Beĭ84]

$$\mathbf{r}_{\mathbb{Z}}^{\text{Beil}} : \pi_*(\mathbf{K}_{\mathbb{Z}}(X)) \otimes \mathbb{R} \rightarrow H^{-*}(\mathbf{DR}_{\mathbb{Z}}(X))$$

is an isomorphism in degrees $* \geq 2$, and injective (with well known image) in degrees 0 and 1. In fact, the conditions 1. – 3. below determine a subcomplex of $\mathbf{DR}_{\mathbb{Z}}(X)$ with trivial differential such that $\pi_*(\mathbf{K}_{\mathbb{Z}}(X)) \otimes \mathbb{R}$ is isomorphic to (the cohomology of) this subcomplex via $\mathbf{r}_{\mathbb{Z}}^{\text{Beil}}$. It follows that for any smooth manifold M we get a natural isomorphism

$$\Psi : \Omega A(M) \hookrightarrow \mathbf{DR}_{Mf, \mathbb{Z}}(M \times X) \quad (91)$$

onto the subcomplex of forms $\omega \in \mathbf{DR}_{Mf, \mathbb{Z}}(M \times X)$ satisfying (using the components introduced in (89))

1. $\omega(p) = 0$, $\omega_{\mathbb{R}}(p) = 0$, $\text{Re}(\tilde{\omega}(p)) = 0$ for $p \geq 2$
2. $\omega(1) = 0$, $\omega_{\mathbb{R}}(1) = 0$, $\text{Re}(\tilde{\omega}(1)) = 0$ and

$$\frac{1}{|X(\mathbb{C})|} \sum_{\sigma \in X(\mathbb{C})} \text{ev}_{\sigma} \tilde{\omega}(1) = 0 . \quad (92)$$

3. $\tilde{\omega}(0) = 0$ and there exists a form $x \in A_{\mathbb{R}}(M)$ such that $\text{ev}_{\sigma} \omega(0) = \text{ev}_{\sigma} \omega_{\mathbb{R}}(0) = x$ for all $\sigma \in X(\mathbb{C})$.

The isomorphism is normalized in the unique manner such that

$$\begin{array}{ccc}
\mathrm{Sm}(KR) & \xrightarrow{\mathrm{rat}, (85)} & H(\Omega A) \\
\downarrow \sim & & \downarrow \Psi \\
p_X^* \mathbf{K}_{Mf, \mathbb{Z}} & \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\mathrm{Beil}}} & p_X^* H(\mathbf{DR}_{Mf, \mathbb{Z}})
\end{array} \quad . \quad (93)$$

commutes. In fact, Ψ is fixed by the evaluation of this diagram at $M = *$ using the fact that rat induces an isomorphism after tensoring its domain by \mathbb{R} .

Lemma 5.10. *The map*

$$\Psi : H^*(\Omega A) \rightarrow p_X^* H^*(\mathbf{DR}_{Mf, \mathbb{Z}})$$

is injective.

Proof. Let M be a smooth manifold. Let $\alpha \in Z^*(\Omega A(M))$ and assume that $\Psi(\alpha) = d\omega$. We must show that there exists some $\hat{\beta} \in \Omega A^{*-1}(M)$ such that $d\hat{\beta} = \alpha$. We define $\beta \in \mathbf{DR}_{Mf, \mathbb{Z}}^{*-1}(M \times X)$ as follows:

1. For $p \geq 2$ we set $\tilde{\beta} := \mathrm{Im}(\tilde{\omega}(p))$ and $\beta(p) := 0$, $\beta_{\mathbb{R}}(p) := 0$.
2. For $p = 1$ we define $\beta(1) := 0$, $\beta_{\mathbb{R}}(1) := 0$ and $\tilde{\beta}(1)$ by

$$\mathrm{ev}_{\sigma} \tilde{\beta}(1) := \mathrm{ev}_{\sigma} \mathrm{Im}(\tilde{\omega}(1)) - \frac{1}{|X(\mathbb{C})|} \sum_{\sigma \in X(\mathbb{C})} \mathrm{ev}_{\sigma} \mathrm{Im}(\tilde{\omega}(1))$$

for all $\sigma \in X(\mathbb{C})$.

3. For $p = 0$ we set $\beta_{\mathbb{R}}(0) := \omega_{\mathbb{R}}(0)$, $\beta(0) := \omega_{\mathbb{R}}(0)$ and $\tilde{\beta}(0) := 0$.

Then we have $d\beta = \Psi(\alpha)$ and since β satisfies 1. – 3. above $\beta = \Psi(\hat{\beta})$ for some uniquely determined $\hat{\beta} \in \Omega A^{*-1}(M)$. We conclude that $d\hat{\beta} = \alpha$. \square

In view of the Definition 5.1 of $\mathrm{Diff}(\mathbf{K})$ the map of diagrams

$$\begin{array}{ccccc}
\mathrm{Sm}(KR) & \xrightarrow{\mathrm{rat}} & H(\Omega A) & \longleftarrow & H(\sigma^{\geq 0} \Omega A) \\
\downarrow \sim & & \downarrow \Psi & & \downarrow \sigma^{\geq 0} \Psi \\
p_X^* \mathbf{K}_{Mf, \mathbb{Z}} & \xrightarrow{\mathbf{r}_{\mathbb{Z}}^{\mathrm{Beil}}} & p_X^* H(\mathbf{DR}_{Mf, \mathbb{Z}}) & \longleftarrow & p_X^* H(\sigma^{\geq 0} \mathbf{DR}_{Mf, \mathbb{Z}})
\end{array}$$

gives a natural transformation $\mathrm{Diff}(KR)(\dots) \rightarrow \mathrm{Diff}(\mathbf{K}(X))$ and hence a natural transformation

$$\psi : \widehat{KR}^0 \rightarrow \widehat{\mathbf{K}}(X)^0 .$$

By construction the following diagram commutes:

$$\begin{array}{ccccc}
& & & I & \\
& & \Omega A^{-1} \xrightarrow{a} \widehat{KR}^0 & \xrightarrow{R} & Z^0(\Omega A) & \xrightarrow{\quad} & KR^0 \\
& \downarrow \Psi & \downarrow \psi & & \downarrow \Psi & & \downarrow \cong \\
p_X^* \mathbf{DR}_{Mf, \mathbb{Z}}^{-1} & \xrightarrow{a} & \widehat{\mathbf{K}}(X)^0 & \xrightarrow{R} & Z^0(p_X^* \mathbf{DR}_{Mf, \mathbb{Z}}) & \xrightarrow{\quad} & \mathbf{K}_{\mathbb{Z}}(X)^0 \\
& & & I & & &
\end{array} \quad (94)$$

Lemma 5.11. *For every manifold $M \in \mathbf{Mf}$ the map*

$$\psi : \widehat{KR}^0(M) \rightarrow \widehat{\mathbf{K}}(X)^0(M)$$

is injective.

Proof. Compare the exact sequence (68) with the analogous exact sequence for \widehat{KR}^0 . The assertion follows by a simple diagram chase using Lemma 5.10. \square

5.5.3 Comparison of cycle maps

A sheaf of locally free, locally finitely generated $\mathbf{pr}_X^* \mathcal{O}_X$ -modules V on $M \times X$ is the same as a locally constant sheaf of finitely generated projective R -modules on M . It gives rise to complex vector bundles $V_\sigma \rightarrow M$ for all $\sigma \in X(\mathbb{C})$. Since $X(\mathbb{C})$ is zero-dimensional the datum of a geometric bundle (V, g^V) on $M \times X$ reduces to the choice of hermitean metrics h^{V_σ} on the complex vector bundles V_σ for all $\sigma \in X(\mathbb{C})$ such that $h^{V_\sigma} = \bar{h}^{V_{\bar{\sigma}}}$. This is the same as a geometry on the locally constant sheaf of finitely generated projective R -modules on M as considered in [BG, Def. 3.8]. We therefore have a natural identification of the domains of the cycle maps

$$\pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}}^{geom}(M \times X)) \cong \overline{\mathbf{Loc}}_{geom}^{proj}(M)$$

(see 4.18 and [BG, Sec. 3.6], respectively). We let

$$\widehat{\mathbf{cyc}} : \overline{\mathbf{Loc}}_{geom}^{proj}(M) \rightarrow \widehat{KR}^0(M)$$

and

$$\widehat{\mathbf{cyc}} : \pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}}^{geom}(M \times X)) \rightarrow \widehat{\mathbf{K}}^0(M \times X)$$

denote the corresponding cycle maps (see Definition 5.5 for the latter). As the comparison in Lemma 5.14 shows these are not equal, but differ by a natural correction term.

In the present paper we let $\beta(g^V) \in Z^0(\Omega A(M))$ denote the characteristic form of the geometric bundle (V, g) as introduced in [BG, Def. 3.9], and $\omega(g^V) \in Z^0(\mathbf{DR}_{Mf, \mathbb{Z}}(M \times$

X)) be the form introduced in Definition 4.13. In order to simplify formulas we use a normalization adapted to the conventions of the present paper. We write (cf. (89))

$$\Psi(\beta(g^V)) = \left(\beta(g^V)_{\mathbb{R}}(p) \oplus \beta(g^V)(p), \tilde{\beta}(g^V)(p) \right)_{p \geq 0} . \quad (95)$$

Then the characteristic form $\beta(g^V)$ is determined by

1. For $p \geq 2$ we have $\beta(g^V)_{\mathbb{R}}(p) = 0 = \beta(g^V)(p)$, and for all $\sigma \in X(\mathbb{C})$

$$\text{ev}_{\sigma} \tilde{\beta}(g^V)(p) = \frac{1}{2} \tilde{\mathbf{ch}}_{2p-1}(\nabla^{V_{\sigma,*}}, \nabla^{V_{\sigma}}) .$$

2. If $p = 1$, then $\beta(g^V)_{\mathbb{R}}(1) = 0 = \beta(g^V)(1)$, and for all $\sigma \in X(\mathbb{C})$

$$\text{ev}_{\sigma} \tilde{\beta}(g^V)(1) = \frac{1}{2} \tilde{\mathbf{ch}}_1(\nabla^{V_{\sigma,*}}, \nabla^{V_{\sigma}}) - \kappa ,$$

where

$$\kappa := \frac{1}{2|X(\mathbb{C})|} \sum_{\sigma \in X(\mathbb{C})} \tilde{\mathbf{ch}}_1(\nabla^{V_{\sigma,*}}, \nabla^{V_{\sigma}}) .$$

3. We have $\tilde{\beta}(g^V)(0) = 0$, and for all $\sigma \in X(\mathbb{C})$

$$\text{ev}_{\sigma}(\beta(g^V)_{\mathbb{R}}(0)) = \text{ev}_{\sigma}(\beta(g^V)(0)) := \dim(V) .$$

The following two lemmas prepare the definition of the correction term in the comparison of cycle maps.

Lemma 5.12. *There exists a smooth map $g^V \mapsto \lambda(g^V) \in A_{\mathbb{R}}^0(M)$ which is natural with respect to pull-back along maps between manifolds and such that*

$$\sum_{\sigma \in X(\mathbb{C})} \tilde{\mathbf{ch}}_1(\nabla^{V_{\sigma,*}}, \nabla^{V_{\sigma}}) = d\lambda(g^V) .$$

Proof. Recall that

$$\tilde{\mathbf{ch}}_1(\nabla^{V_{\sigma,*}}, \nabla^{V_{\sigma}}) = \int_{I \times M/M} \mathbf{ch}_2(\nabla^{\tilde{V}_{\sigma}})$$

where $\tilde{V}_{\sigma} \rightarrow I \times M$ is the bundle $\mathbf{pr}_M^* V_{\sigma}$ with the connection $\nabla^{\tilde{V}_{\sigma}}$ obtained by linear interpolation between $\nabla^{V_{\sigma}}$ and $\nabla^{V_{\sigma,*}}$. We consider the line bundle

$$\tilde{\mathbf{V}} := \bigotimes_{\sigma \in X(\mathbb{C})} \det(\tilde{V}_{\sigma})$$

with the connection $\nabla^{\tilde{\mathbf{V}}}$ induced by the connections $\nabla^{\tilde{V}_{\sigma}}$. Then

$$\sum_{\sigma \in X(\mathbb{C})} \tilde{\mathbf{ch}}_1(\nabla^{V_{\sigma,*}}, \nabla^{V_{\sigma}}) = \int_{I \times M/M} \mathbf{ch}_2(\nabla^{\tilde{\mathbf{V}}}) .$$

We further consider the complex line bundle $\mathbf{V} := \bigotimes_{\sigma \in X(\mathbb{C})} \det(V_\sigma)$ on M with the induced flat connection $\nabla^{\mathbf{V}}$. We let $(s, t) \in I \times I$ denote the parameters. On $I \times I \times M$ we consider the bundle $\hat{\mathbf{V}} := \text{pr}_M^* \mathbf{V}$ with the connection $\nabla^{\hat{\mathbf{V}}}$ which along the s -direction linearly interpolates between the connection $\nabla^{\hat{\mathbf{V}}}$ and the linear path (parametrized by t) between $\text{pr}_M^* \nabla^{\mathbf{V}}$ and $\text{pr}_M^* \nabla^{\mathbf{V},*}$. We set

$$\lambda_0(g^V) := - \int_{I \times I \times M/M} \mathbf{ch}_2(\nabla^{\hat{\mathbf{V}}})$$

and observe that

$$d\lambda_0(g^V) = \sum_{\sigma \in X(\mathbb{C})} \tilde{\mathbf{ch}}_1(\nabla^{V_{\sigma,*}}, \nabla^{V_{\sigma}}) - \tilde{\mathbf{ch}}_1(\nabla^{\mathbf{V},*}, \nabla^{\mathbf{V}}).$$

We now observe that \mathbf{V} has a \mathbb{Z} -structure and therefore has a uniquely determined hermitean metric $h_0^{\mathbf{V}}$ such that the \mathbb{Z} -basis vectors have length one. On the bundle $\hat{\mathbf{V}} \rightarrow I \times I \times M$ we consider the metric

$$\hat{h}^{\hat{\mathbf{V}}} := (1-s)\text{pr}_M^* h^{\mathbf{V}} + s\text{pr}_M^* h_0^{\mathbf{V}}.$$

We consider the connection $\hat{\nabla}^{\hat{\mathbf{V}}}$ which linearly (in t) interpolates between the connection $\text{pr}_M^* \nabla^{\mathbf{V}}$ and its adjoint with respect to $\hat{h}^{\hat{\mathbf{V}}}$. We define

$$\lambda_1(g^V) := - \int_{I \times I \times M/M} \mathbf{ch}_2(\hat{\nabla}^{\hat{\mathbf{V}}})$$

and observe that

$$d\lambda_1(g^V) = \tilde{\mathbf{ch}}_1(\nabla^{\mathbf{V},*}, \nabla^{\mathbf{V}}).$$

Since $\text{Im} \sum_{\sigma \in X(\mathbb{C})} \tilde{\mathbf{ch}}_1(\nabla^{V_{\sigma,*}}, \nabla^{V_{\sigma}}) = 0$ (see (90) for Im and Re)

$$\lambda(g^V) := \text{Re}(\lambda_0(g^V) + \lambda_1(g^V)) \tag{96}$$

has the required properties. \square

Let $V \rightarrow M$ be a complex vector bundle and ∇_i , $i = 0, 1, 2$, be three connections on V . Then we have the following well-known relation

$$\tilde{\mathbf{ch}}(\nabla_0, \nabla_1) + \tilde{\mathbf{ch}}(\nabla_1, \nabla_2) \equiv \tilde{\mathbf{ch}}(\nabla_0, \nabla_2)$$

modulo exact forms. In fact, we have a stronger result.

Lemma 5.13. *There exists a smooth map*

$$(\nabla_0, \nabla_1, \nabla_2) \mapsto \tilde{\mathbf{ch}}(\nabla_0, \nabla_1, \nabla_2) \in A(M),$$

natural with respect to pull-back along maps between manifolds, such that

$$\tilde{\mathbf{ch}}(\nabla_0, \nabla_1) + \tilde{\mathbf{ch}}(\nabla_1, \nabla_2) - \tilde{\mathbf{ch}}(\nabla_0, \nabla_2) = d\tilde{\mathbf{ch}}(\nabla_0, \nabla_1, \nabla_2).$$

Proof. We consider the bundle $\hat{V} := \mathbf{pr}_M^* V \rightarrow \Delta^2 \times M$ and define the connection $\nabla^{\hat{V}}$ as the convex linear interpolation between the connections ∇_i at the corners of the simplex. Then

$$\tilde{\mathbf{ch}}(\nabla_0, \nabla_1, \nabla_2) := \int_{\Delta^2 \times M/M} \mathbf{ch}(\nabla^{\hat{V}})$$

does the job. \square

We now define the natural transformation that will show up in the above mentioned correction term

$$\alpha : \overline{\mathbf{Loc}}_{geom}^{proj}(M) \rightarrow \mathbf{DR}_{Mf, \mathbb{Z}}^{-1}(M \times X) , \quad (V, g^V) \mapsto \alpha(g^V) . \quad (97)$$

We use the notation (89) in order to describe the form $\alpha(g^V)$:

1. For $p \geq 2$ we set

$$\mathbf{ev}_\sigma \alpha(g^V)_{\mathbb{R}}(p) := \frac{1}{2} \left(\tilde{\mathbf{ch}}_{2p-1}(\nabla^{V_\sigma, u}, \nabla^{V_\sigma}) + \tilde{\mathbf{ch}}_{2p-1}(\nabla^{V_\sigma, u}, \nabla^{V_\sigma, *}) \right) ,$$

$$\mathbf{ev}_\sigma \tilde{\alpha}(g^V)(p) := -\frac{1}{2} \tilde{\mathbf{ch}}_{2p-2}(\nabla^{V_\sigma, *}, \nabla^{V_\sigma, u}, \nabla^{V_\sigma}) ,$$

and $\alpha(g^V)(p) := 0$.

2. For $p = 1$ we set (using (96))

$$\mathbf{ev}_\sigma \alpha(g^V)_{\mathbb{R}}(1) := \frac{1}{2} \left(\tilde{\mathbf{ch}}_{2p-1}(\nabla^{V_\sigma, u}, \nabla^{V_\sigma}) + \tilde{\mathbf{ch}}_{2p-1}(\nabla^{V_\sigma, u}, \nabla^{V_\sigma, *}) \right) ,$$

$$\mathbf{ev}_\sigma \tilde{\alpha}(g^V)(1) := -\frac{1}{2} \tilde{\mathbf{ch}}_0(\nabla^{V_\sigma, *}, \nabla^{V_\sigma, u}, \nabla^{V_\sigma}) + \frac{1}{2|X(\mathbb{C})|} \lambda(g^V) ,$$

and $\alpha(g^V)(1) := 0$.

3. We set $\tilde{\alpha}(g^V)(0) := 0$, $\alpha(g^V)_{\mathbb{R}}(0) := 0$ and $\alpha(g^V)(0) := 0$

A simple calculation shows that

$$\omega(g^V) = \Psi(\beta(g^V)) + d\alpha(g^V) . \quad (98)$$

In order to compare the two cycle maps

$$\widehat{\mathbf{cyc}l}, \psi \circ \hat{\mathbf{cyc}l} : \overline{\mathbf{Loc}}_{geom}^{proj} \cong p_X^* \pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}}^{geom}) \rightarrow \hat{\mathbf{K}}(X)^0$$

we introduce the following notations. We have a natural forgetful map

$$V : p_X^* \pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}}^{geom}) \rightarrow p_X^* \pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}}).$$

We denote the flat part (66) of differential algebraic K -theory by

$$\widehat{\mathbf{K}}(X)_{flat}^0(M) := \ker \left(\widehat{\mathbf{K}}^0(M \times X) \xrightarrow{R} Z^0(\mathbf{DR}_{Mf, \mathbb{Z}}(M \times X)) \right).$$

An important observation is that $\widehat{\mathbf{K}}(X)_{flat}^0$ is, in contrast to $\widehat{\mathbf{K}}(X)^0$, part of a cohomology theory. To be precise, $\widehat{\mathbf{K}}(X)_{flat}$ is represented by the homotopy fibre of the map of spectra $\mathbf{r}_{\mathbb{Z}}^{\text{Beil}} : \mathbf{K}_{\mathbb{Z}}(X) \rightarrow H(\mathbf{DR}_{\mathbb{Z}}(X))$. In particular, it makes sense to evaluate $\widehat{\mathbf{K}}(X)_{flat}^0$ on any topological space.

Lemma 5.14. *There exists a natural transformation*

$$\epsilon : p_X^* \pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}}) \rightarrow \widehat{\mathbf{K}}(X)_{flat}^0$$

such that

$$\psi \circ \widehat{\text{cyc}}\mathbf{1} - \widehat{\text{cyc}}\mathbf{1} + a \circ \alpha = \epsilon \circ V.$$

Moreover, ϵ is additive on short exact sequences, i.e. if

$$0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0 \tag{99}$$

is an exact sequence then $\epsilon(V_1) = \epsilon(V_0) + \epsilon(V_2)$.

Proof. We consider the natural transformation

$$\delta := \psi \circ \widehat{\text{cyc}}\mathbf{1} - \widehat{\text{cyc}}\mathbf{1} + a \circ \alpha : p_X^* \pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}}^{geom}) \rightarrow \widehat{\mathbf{K}}(X)^0.$$

The equality of characteristic forms (98) implies that $R \circ \delta = 0$. Since we can interpolate between any two geometries on a given bundle the homotopy formula 5.6 implies that δ factors as a composition

$$p_X^* \pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}}^{geom}) \xrightarrow{V} p_X^* \pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}}) \xrightarrow{\epsilon} \widehat{\mathbf{K}}(X)_{flat}^0 \subseteq \widehat{\mathbf{K}}(X)^0.$$

Since δ and hence ϵ is additive for direct sums and vanishes on a trivial bundle it corresponds to a class $\epsilon \in \widehat{\mathbf{K}}(X)_{flat}^0(BGL(R))$ where $BGL(R)$ is the classifying space of the discrete group $GL(R)$. Since $\widehat{\mathbf{K}}(X)_{flat}^0$ is part of a cohomology theory it does not change by passing to the plus-construction $\widehat{\mathbf{K}}(X)_{flat}^0(BGL(R)^+) \xrightarrow{\cong} \widehat{\mathbf{K}}(X)_{flat}^0(BGL(R))$.

We denote by $c_V : M \rightarrow BGL(R)$ the map that classifies the stable class of $V \in p_X^* \pi_0(i\mathbf{Vect}_{Mf, \mathbb{Z}})(M)$. Given an extension (99) one knows that c_{V_1} and $c_{V_0 \oplus V_2}$ are homotopic when composed with $BGL(R) \rightarrow BGL(R)^+$. This implies the assertion of the Lemma. \square

5.6 Extensions

The main result of this Subsection is the proof (5.6.4) of Theorem 1.2, which identifies the alternating sum of cycle classes for an extension of geometric vector bundles with a version of the Bismut-Lott higher torsion form \mathcal{T} (Lott's relation). We refer to 5.6.3 for the precise statement. The main ingredient is a deformation to a split extension over \mathbb{CP}^1 . Using integration in the algebraic direction (5.4.2) we construct in 5.6.2 a form \mathcal{T}' and show that it coincides with the Bismut-Lott form \mathcal{T} up to exact forms. For this we use the axiomatic characterization of the latter, to be recalled in 5.6.1 below. Lott's relation then follows essentially from the homotopy formula in Proposition 5.8.

5.6.1 The axiomatic characterization of the torsion form

We specialize the definitions of geometries 4.7 and 4.8 to the case $X = *$. A locally constant sheaf V of finitely generated \mathbb{C} -vector spaces on a manifold M can be presented as the sheaf of parallel sections of a uniquely determined flat complex vector bundle (V, ∇^V) on M . The connection ∇^V is a partial geometry on that bundle in the sense of Definition 4.7. A geometry on V in the sense of Definition 4.8 is a hermitean metric h^V on the vector bundle $V \rightarrow M$. In this case $\nabla^{V,*}$ denotes the adjoint connection.

Let

$$\mathcal{V} : 0 \rightarrow V_0 \rightarrow \cdots \rightarrow V_n \rightarrow 0$$

be an exact sequence of locally constant sheaves of finitely generated \mathbb{C} -vector spaces on a manifold M . A geometry on \mathcal{V} is by definition a collection $h^\mathcal{V} = (h^{V_i})_{i=0,\dots,n}$ of metrics on the bundles V_i . In this situation the Bismut-Lott torsion form

$$\mathcal{T}(\mathcal{V}, h^\mathcal{V}) \in A(M)$$

is defined such that

$$d\mathcal{T}(\mathcal{V}, h^\mathcal{V}) = \frac{1}{2} \sum_{i=0}^n (-1)^n \tilde{\mathbf{ch}}(\nabla^{V_i,*}, \nabla^{V_i}) \quad (100)$$

[BL95, Def. 2.20]. Note that our normalization of the torsion form differs from the one adopted in [BL95]. In the present paper we will not use the precise construction of the torsion form but rather its axiomatic characterization. We restrict to the case $n = 2$ and consider a transformation

$$(\mathcal{V}, h^\mathcal{V}) \mapsto \mathcal{T}'(\mathcal{V}, h^\mathcal{V})$$

which associates to every exact sequence

$$\mathcal{V} : 0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0 \quad (101)$$

of locally constant sheaves of finitely generated \mathbb{C} -vector spaces on a manifold M equipped with a geometry $h^\mathcal{V}$ a form $\mathcal{T}'(\mathcal{V}, h^\mathcal{V}) \in A(M)$. We have the following characterization due to Bismut-Lott:

Theorem 5.15 ([BL95, Thm. A1.2]). *Assume that $(\mathcal{V}, h^\mathcal{V}) \mapsto \mathcal{T}'(\mathcal{V}, h^\mathcal{V})$ satisfies*

1. $\text{Re}(\mathcal{T}'(\mathcal{V}, h^\mathcal{V})_{2p-2}) = 0$ for all $p \geq 1$ (see (90) for Re) ,
2. $d\mathcal{T}'(\mathcal{V}, h^\mathcal{V}) = \frac{1}{2} \sum_{i=0}^2 (-1)^i \tilde{\mathbf{ch}}(\nabla^{V_i, *}, \nabla^{V_i})$,
3. \mathcal{T}' is natural, i.e. $\mathcal{T}'(f^*\mathcal{V}, f^*h^\mathcal{V}) = f^*\mathcal{T}'(\mathcal{V}, h^\mathcal{V})$ for a smooth map $f : M' \rightarrow M$,
4. $\mathcal{T}'(\mathcal{V}, h^\mathcal{V}) = 0$ in the split case, i.e. when $V_1 \cong V_0 \oplus V_2$ as flat bundles and $h^{V_1} = h^{V_0} \oplus h^{V_2}$,
5. \mathcal{T}' depends smoothly on \mathcal{V} and $h^\mathcal{V}$.

Then $\mathcal{T}'(\mathcal{V}, h^\mathcal{V})$ equals the Bismut-Lott torsion form $\mathcal{T}(\mathcal{V}, h^\mathcal{V})$ modulo exact forms:

$$\mathcal{T}'(\mathcal{V}, h^\mathcal{V}) \equiv \mathcal{T}(\mathcal{V}, h^\mathcal{V}) \in A(M)/\text{im}(d) .$$

5.6.2 A deformation

In this subsection we give a holomorphic construction of a version of the analytic torsion form. Let $(\mathcal{V}, h^\mathcal{V})$ be an exact sequence (101) with geometry. We construct a form $\mathcal{T}'(\mathcal{V}, h^\mathcal{V})$ which satisfies the assumptions in Theorem 5.15. The main point of the construction below is that it is canonical. This ensures the properties 3. and 5. in Theorem 5.15.

The metric h^{V_1} provides an orthogonal decomposition of vector bundles $V_1 \cong V_0 \oplus V_2$. With respect to this decomposition the connection on V_1 can be written as

$$\nabla^{V_1} = \nabla^{V_0} \oplus \nabla^{V_2} + E , \quad E \in A^1(M, \text{Hom}(V_2, V_0)) .$$

Furthermore,

$$h^{V_1} = h_{|V_0}^{V_1} \oplus h_{|V_2}^{V_1} .$$

We let $Y := \mathbb{CP}^1$ with homogeneous coordinates $[x : y]$. We consider x, y as global sections in $\mathcal{O}_Y(1)(Y)$. The holomorphic line bundle $H_Y(1)$ (whose sheaf of holomorphic sections is $\mathcal{O}_Y(1)$) has a natural $U(2)$ -invariant metric $h^{H_Y(1)}$ and an $U(2)$ -invariant connection. We normalize $h^{H_Y(1)}$ in such a way that its value on the section y evaluated at $[1, 1]$ is 1. On $M \times Y$ we consider the vector bundle

$$W := \text{pr}_M^* V_0 \otimes \text{pr}_Y^* H_Y(1) \oplus \text{pr}_M^* V_2 .$$

The connections ∇^{V_0} and ∇^{V_2} together with the connection on $H_Y(1)$ induce a connection ∇_0^W on W . We consider

$$\text{pr}_M^* E \otimes \text{pr}_Y^* x \in A^1(M \times Y, \text{Hom}(\text{pr}_M^* V_2, \text{pr}_M^* V_0 \otimes \text{pr}_Y^* H_Y(1)))$$

and define the connection

$$\nabla^W := \nabla_0^W + \text{pr}_M^*(E) \otimes \text{pr}_Y^* x .$$

The partial connection ∇^I on $W \rightarrow M \times Y$ induced by ∇^W in the M -direction is flat. Furthermore, the $(0, 1)$ -part $\bar{\partial}$ of ∇^W in the Y -direction is a holomorphic structure. The pair $(\nabla^I, \bar{\partial})$ is a partial geometry on the complex vector bundle $W \rightarrow M \times Y$ in the sense of Definition 4.7.

We fix a cut-off function $\theta \in C^\infty(\mathbb{R})$ such that $\theta(t) \in [0, 1]$ for all $t \in \mathbb{R}$, $\theta(t) \equiv 0$ for $t \leq \frac{1}{4}$ and $\theta(t) \equiv 1$ for $t \geq \frac{3}{4}$. The bundle W has two metrics h_{02}^W and h_1^W , where the first one is induced from h^{V_0} , h^{V_2} and $h^{H_Y(1)}$, while the second one is induced in the same manner from $h_{|V_0}^{V_1}$ and $h_{|V_2}^{V_1}$ and $h^{H_Y(1)}$. We define the metric

$$h^W([x : 1]) := (1 - \theta(|x|))h_{02}^W([x : 1]) + \theta(|x|)h_1^W([x : 1])$$

(note that h^W extends smoothly to the point $[1 : 0] \in Y$).

The pair (∇^{II}, h^W) consisting of the partial connection ∇^{II} in the Y -direction induced by ∇^W and the metric h^W is a geometry g^W in the sense of Definition 4.8, and the pair (W, g^W) becomes a geometric bundle. In particular, we have the characteristic form (54)

$$\omega(g^W) \in Z^0(\mathbf{DR}_{Mf, \mathbb{C}}(M \times Y)) .$$

Using (81) we define the form

$$\mathcal{U}(\mathcal{V}, h^\mathcal{V}) := \int_{II} \omega(g^W) \in \mathbf{DR}_{Mf, \mathbb{C}}^{-1}(M) . \quad (102)$$

Let $u_i : M \hookrightarrow M \times \mathbb{CP}^1$ be the inclusions given by the points $[i, 1] \in \mathbb{CP}^1$, $i = 0, 1$. Then by construction and using y as a basis of $\mathcal{O}_Y(1)_{|\{y \neq 0\}}$ we have isomorphisms

$$u_1^*(W, g^W) \cong (V_1, h^{V_1}) , \quad u_0^*(W, g^W) \cong (V_0 \oplus V_2, h^{V_0} \oplus h^{V_2}) .$$

Since $\omega(g^W)$ is closed (82) implies that

$$-d\mathcal{U}(\mathcal{V}, h^\mathcal{V}) = \sum_{i=0}^2 (-1)^i \omega(h^{V_i}) =: \omega(h^\mathcal{V}) . \quad (103)$$

We define the form $\alpha(h^V) \in \mathbf{DR}_{Mf, \mathbb{C}}^{-1}(M)$ using the notation (89) similarly to the form defined in (97):

1. For $p \geq 1$ we set

$$\alpha(h^V)_{\mathbb{R}}(p) := \frac{1}{2} \left(\tilde{\mathbf{c}}_{2p-1}(\nabla^{V,u}, \nabla^V) + \tilde{\mathbf{c}}_{2p-1}(\nabla^{V,u}, \nabla^{V,*}) \right) ,$$

$$\tilde{\alpha}(h^V)(p) := -\frac{1}{2} \tilde{\mathbf{c}}_{2p-2}(\nabla^{V,*}, \nabla^{V,u}, \nabla^V) ,$$

and $\alpha(h^V)(p) := 0$.

2. We set $\tilde{\alpha}(h^V)(0) := 0$, $\alpha(h^V)_{\mathbb{R}}(0) := 0$ and $\alpha(g^V)(0) := 0$.

We set

$$\alpha(h^\mathcal{V}) := \sum_{i=0}^2 (-1)^i \alpha(h^{V_i}) \in \mathbf{DR}_{Mf,\mathbb{C}}^{-1}(M) . \quad (104)$$

Furthermore, we define

$$\gamma(h^\mathcal{V}) := \sum_{i=0}^2 (-1)^i \gamma(h^{V_i}) \in Z^0(\mathbf{DR}_{Mf,\mathbb{C}}(M)) ,$$

where $\gamma(h^V) \in Z^0(\mathbf{DR}_{Mf,\mathbb{C}}(M))$ is given as follows:

1. For $p \geq 1$ we set $\gamma(h^V)_\mathbb{R}(p) := 0$, $\gamma(h^V)(p) := 0$ and

$$\tilde{\gamma}(h^V)(p) := \frac{1}{2} \tilde{\mathbf{ch}}_{2p-1}(\nabla^{V,*}, \nabla^V) .$$

2. We set $\tilde{\gamma}(h^V)(0) := 0$, $\gamma(h^V)_\mathbb{R}(0) = \gamma(h^V)(0) := \dim(V)$.

Then we have (compare with (98))

$$\omega(h^\mathcal{V}) - d\alpha(h^\mathcal{V}) = \gamma(h^\mathcal{V}) . \quad (105)$$

We define

$$\mathcal{Z}(\mathcal{V}, h^\mathcal{V}) := \mathcal{U}(\mathcal{V}, h^\mathcal{V}) + \alpha(h^\mathcal{V}) \in \mathbf{DR}_{Mf,\mathbb{C}}^{-1}(M) . \quad (106)$$

Equations (103) and (105) imply that its components (89) satisfy $\mathcal{Z}(\mathcal{V}, h^\mathcal{V})(p) = 0$ and

$$d\mathcal{Z}_\mathbb{R}(\mathcal{V}, h^\mathcal{V})(p) = 0 , \quad -d\tilde{\mathcal{Z}}(\mathcal{V}, h^\mathcal{V})(p) + \mathcal{Z}_\mathbb{R}(\mathcal{V}, h^\mathcal{V})(p) = -\tilde{\gamma}(h^\mathcal{V})(p) \quad (107)$$

for all $p \geq 0$. We now separate the real and imaginary parts (cf. (90)) of this equality. Note that

$$\mathrm{Im}(\mathcal{Z}_\mathbb{R}(\mathcal{V}, h^\mathcal{V})(p)) = 0 , \quad \mathrm{Re}(\tilde{\mathbf{ch}}_{2p-1}(\nabla^{V_i,*}, \nabla^{V_i})) = 0 .$$

We define

$$\mathcal{T}'(\mathcal{V}, h^\mathcal{V})_{2p-2} := \mathrm{Im}(\tilde{\mathcal{Z}}(\mathcal{V}, h^\mathcal{V})(p)) , \quad \mathcal{T}'(\mathcal{V}, h^\mathcal{V}) := \sum_{p \geq 1} \mathcal{T}'(\mathcal{V}, h^\mathcal{V})_{2p-2} .$$

Then the imaginary part of (107) gives

$$d\mathcal{T}'(\mathcal{V}, h^\mathcal{V}) = \frac{1}{2} \sum_{i=0}^2 (-1)^i \tilde{\mathbf{ch}}(\nabla^{V_i,*}, \nabla^{V_i}) . \quad (108)$$

Finally, we check that $\mathcal{T}'(\mathcal{V}, h^\mathcal{V})$ vanishes in the split case. But indeed, using the notations from the beginning of this subsection, the metric $h^W = h_{02}^W = h_1^W$, the connection $\nabla^W = \nabla_0^W$, and hence the characteristic forms are $U(2)$ -invariant. In particular, $\omega(g^W) \in \mathbf{DR}_{Mf,\mathbb{C}}(M \times Y)$ is harmonic in the Y -direction and, by construction (cf. (83),(84)), the integral $\int_{II} \omega(g^W)$ vanishes. One sees directly that also $\tilde{\alpha}(h^\mathcal{V})$ vanishes.

Putting everything together, we see that

$$(\mathcal{V}, h^\mathcal{V}) \mapsto \mathcal{T}'(\mathcal{V}, h^\mathcal{V})$$

satisfies the assumptions in Theorem 5.15 characterizing the Bismut-Lott torsion form, and hence

$$\mathcal{T}'(\mathcal{V}, h^\mathcal{V}) \equiv \mathcal{T}(\mathcal{V}, h^\mathcal{V}) \quad (109)$$

modulo exact forms.

For later use we introduce the following notation:

$$\mathcal{Z}^{\text{Im}}(\mathcal{V}, h^\mathcal{V}) := (0 \oplus 0, \mathcal{T}'(\mathcal{V}, h^\mathcal{V})_{2p-2})_{p \geq 0} \quad (110)$$

and

$$\mathcal{Z}^{\text{Re}}(\mathcal{V}, h^\mathcal{V}) := \left(\mathcal{Z}_{\mathbb{R}}(\mathcal{V}, h^\mathcal{V})(p) \oplus 0, \text{Re}(\tilde{\mathcal{Z}}(\mathcal{V}, h^\mathcal{V})(p)) \right)_{p \geq 0}$$

so that

$$\mathcal{Z}(\mathcal{V}, h^\mathcal{V}) = \mathcal{Z}^{\text{Re}}(\mathcal{V}, h^\mathcal{V}) + \mathcal{Z}^{\text{Im}}(\mathcal{V}, h^\mathcal{V}) . \quad (111)$$

Further note that (107) implies

$$\mathcal{Z}^{\text{Re}}(\mathcal{V}, h^\mathcal{V}) = d \left(\text{Re}(\tilde{\mathcal{Z}}(\mathcal{V}, h^\mathcal{V})(p)) \oplus 0, 0 \right)_{p \geq 0} . \quad (112)$$

5.6.3 Lott's relation

We fix a number ring R and set $X := \text{Spec}(R) \in \mathbf{Reg}_{\mathbb{Z}}$. We further consider a smooth manifold M and an exact sequence

$$\mathcal{V} : 0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2 \rightarrow 0 \quad (113)$$

of locally free, locally finitely generated $\mathbf{pr}_X^* \mathcal{O}_X$ -modules on $M \times X$.

Recall from Subsection 5.5.3 that for every $\sigma \in X(\mathbb{C})$ we get an exact sequence

$$\mathcal{V}_\sigma : 0 \rightarrow V_{0,\sigma} \rightarrow V_{1,\sigma} \rightarrow V_{2,\sigma} \rightarrow 0 \quad (114)$$

of locally constant sheaves of finitely generated \mathbb{C} -vector spaces. Since $X(\mathbb{C})$ is zero-dimensional, a geometry $g^\mathcal{V} = (g^{V_i})_{i=0,1,2}$ is the same as a collection of geometries $(h^{\mathcal{V}_\sigma})_{\sigma \in X(\mathbb{C})}$ such that $\bar{h}^{\mathcal{V}_\sigma} = h^{\mathcal{V}_\sigma}$ (cf. the introductory paragraphs of 5.5.3 and 5.6.1).

We use the Bismut-Lott torsion forms $\mathcal{T}(\mathcal{V}_\sigma, h^{\mathcal{V}_\sigma})$ in order to define the analytic torsion form

$$\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\mathcal{V}) \in \Omega A^{-1}(M)$$

which coincides up to normalization with the form [BG, (104)]. In terms of the components (89) of its image $\Psi(\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\mathcal{V}))$ under the map (91) it is given by

$$\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\mathcal{V})_{\mathbb{R}}(p) := 0 , \quad \mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\mathcal{V})(p) := 0 \quad \text{for all } p \geq 0$$

and

$$\mathrm{ev}_\sigma \tilde{\mathcal{T}}_{\mathbb{Z}}(\mathcal{V}, g^\mathcal{V})(p) := \begin{cases} -\mathcal{T}(\mathcal{V}_\sigma, h^{\mathcal{V}_\sigma})_{2p-2} & p \geq 2 \\ -\mathcal{T}(\mathcal{V}_\sigma, h^{\mathcal{V}_\sigma})_0 + \bar{\tau}(\mathcal{V}, g^\mathcal{V}) & p = 1 \\ 0 & p = 0 \end{cases} \quad (115)$$

with

$$\bar{\tau}(\mathcal{V}, g^\mathcal{V}) := \frac{1}{|X(\mathbb{C})|} \sum_{\sigma \in X(\mathbb{C})} \mathcal{T}(\mathcal{V}_\sigma, h^{\mathcal{V}_\sigma})_0. \quad (116)$$

Recall the characteristic form $\beta(g^{V_i})$ introduced in (95) and write

$$\beta(g^\mathcal{V}) := \sum_{i=0}^2 (-1)^i \beta(V_i, g^{V_i}).$$

By the fundamental property (100) of the Bismut-Lott torsion form we then have

$$d\Psi(\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\mathcal{V})) = \Psi(\beta(g^\mathcal{V})) \quad \text{and hence} \quad d\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\mathcal{V}) = \beta(g^\mathcal{V})$$

In [BG, Conj. 5.7] we asked whether

$$\mathrm{c\hat{y}cl}(V_0, g^{V_0}) - \mathrm{c\hat{y}cl}(V_1, g^{V_1}) + \mathrm{c\hat{y}cl}(V_2, g^{V_2}) = a(\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\mathcal{V}))$$

holds true in $\widehat{KR}^0(M)$. We call this Lott's relation. For more motivation we refer to [BG, Sec. 3.4.1].

Theorem 5.16. *Lott's relation holds true, i.e. we have*

$$\mathrm{c\hat{y}cl}(V_0, g^{V_0}) - \mathrm{c\hat{y}cl}(V_1, g^{V_1}) + \mathrm{c\hat{y}cl}(V_2, g^{V_2}) = a(\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\mathcal{V}))$$

in $\widehat{KR}^0(M)$.

5.6.4 Proof of Theorem 5.16

Recall the functions (116) and (96).

Lemma 5.17. *We have the relation*

$$\bar{\tau}(\mathcal{V}, g^\mathcal{V}) = \frac{1}{2|X(\mathbb{C})|} \sum_{i=0}^2 (-1)^i \lambda(g_i^V) .$$

Proof. We write

$$\sigma(\mathcal{V}, g^\mathcal{V}) := \bar{\tau}(\mathcal{V}, g^\mathcal{V}) - \frac{1}{2|X(\mathbb{C})|} \sum_{i=0}^2 (-1)^i \lambda(g_i^V) .$$

We first observe that as a consequence of Lemma 5.12 and (108) we have $d\sigma(\mathcal{V}, g^\mathcal{V}) = 0$. This implies that $\sigma(\mathcal{V}, g^\mathcal{V})$ is independent of the choice of the metric $g^\mathcal{V}$. Moreover it

suffices to check the equality $\sigma(\mathcal{V}, g^\nu) = 0$ in the case that M is a point. If M is a point, then we can assume that (\mathcal{V}, g^ν) splits in which case $\sigma(\mathcal{V}, g^\nu) = 0$ is clear. \square

To prove the Theorem, by Lemma 5.11 and (94) it suffices to show that

$$\psi(\widehat{\text{cyc}}\mathbf{l}(V_0, g^{V_0})) - \psi(\widehat{\text{cyc}}\mathbf{l}(V_1, g^{V_1})) + \psi(\widehat{\text{cyc}}\mathbf{l}(V_2, g^{V_2})) = a(\Psi(\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\nu))$$

in $\widehat{\mathbf{K}}^0(M \times X)$. By Lemma 5.14 this is equivalent to

$$\widehat{\text{cyc}}\mathbf{l}(V_0, g^{V_0}) - \widehat{\text{cyc}}\mathbf{l}(V_1, g^{V_1}) + \widehat{\text{cyc}}\mathbf{l}(V_2, g^{V_2}) = a(\Psi(\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\nu)) + \alpha(g^\nu)) ,$$

where $\alpha(g^\nu)$ is defined using the form α from (97) by

$$\alpha(g^\nu) := \sum_{i=0}^2 (-1)^i \alpha(g^{V_i}) .$$

We write

$$\Psi(\mathcal{T}_{\mathbb{Z}}(\mathcal{V}, g^\nu)) + \alpha(g^\nu) =: \delta(\mathcal{V}, g^\nu) \in \mathbf{DR}_{Mf, \mathbb{Z}}^{-1}(M \times X) .$$

Then we have

$$\begin{aligned} \text{ev}_\sigma \delta(\mathcal{V}, g^\nu) &\stackrel{(109), (110), (115)}{=} -\mathcal{Z}^{\text{Im}}(\mathcal{V}_\sigma, h^{\nu_\sigma}) + \alpha(g^{\nu_\sigma}) \pmod{\text{im}(d)} \\ &\stackrel{(111)}{=} -\mathcal{Z}(\mathcal{V}_\sigma, h^{\nu_\sigma}) + \alpha(g^{\nu_\sigma}) + \mathcal{Z}^{\text{Re}}(\mathcal{V}_\sigma, h^{\nu_\sigma}) \\ &\stackrel{(106)}{=} -\mathcal{U}(\mathcal{V}_\sigma, h^{\nu_\sigma}) + \mathcal{Z}^{\text{Re}}(\mathcal{V}_\sigma, h^{\nu_\sigma}) \\ &\stackrel{(112)}{=} -\mathcal{U}(\mathcal{V}_\sigma, h^{\nu_\sigma}) \pmod{\text{im}(d)} \end{aligned} \tag{117}$$

using Lemma 5.17 for the $p = 1$ -component.

We can assume that M is connected and fix a base point $m \in M$. We let $U_i \in \mathbf{Mod}(R)$ be the fibres of V_i at m . The sheaves V_i are determined by the holonomy representations ρ_i of $\pi_1(M, m)$ on the finitely generated projective R -modules U_i . We can choose a decomposition

$$U_1 = U_0 \oplus U_2 \tag{118}$$

as R -modules and write

$$\rho_1 = \begin{pmatrix} \rho_0 & \nu \\ 0 & \rho_2 \end{pmatrix}$$

for some map $\nu : \pi_1(M, m) \rightarrow \mathbf{Hom}_{\mathbf{Mod}(R)}(U_2, U_0)$.

We let $Y := \mathbb{P}_{\mathbb{Z}}^1$ with homogeneous coordinates $[x : y]$, and we consider $x \in \mathcal{O}_Y(1)(Y)$. We define the vector bundle

$$\tilde{U}_1 := U_0 \otimes_R \mathcal{O}_Y(1) \oplus U_2 \otimes_R \mathcal{O}_Y$$

on $X \times Y$ and set

$$\tilde{\rho}_1 := \begin{pmatrix} \rho_0 \otimes 1 & \nu \otimes x \\ 0 & \rho_2 \end{pmatrix} : \pi_1(M, m) \rightarrow \mathbf{Aut}(\tilde{U}_1) .$$

The representation $\tilde{\rho}_1$ determines a sheaf \tilde{V}_1 of locally free, finitely generated $\mathbf{pr}_{X \times Y}^* \mathcal{O}_{X \times Y}$ -modules on $M \times X \times Y$. Its base-change along the inclusion $\mathbb{Z} \rightarrow \mathbb{C}$ is a sheaf $\tilde{V}_{1, \mathbb{C}}$ over $M \times X(\mathbb{C}) \times \mathbb{CP}^1$. For every $\sigma \in X(\mathbb{C})$ we get a complex vector bundle $\tilde{W}_\sigma \rightarrow M \times \mathbb{CP}^1$ with partial geometry $(\nabla^{I, \tilde{W}_\sigma}, \bar{\partial}^{\tilde{W}_\sigma})$ such that the sheaf of joint kernels of $\nabla^{I, \tilde{W}_\sigma}$ and $\bar{\partial}^{\tilde{W}_\sigma}$ is canonically isomorphic to $(\tilde{V}_{1, \mathbb{C}})_{M \times \{\sigma\} \times \mathbb{CP}^1}$.

If we apply the construction of Subsection 5.6.2 to the sequence (114) we get a bundle $W_\sigma \rightarrow M \times \mathbb{CP}^1$ with partial geometry $(\nabla^{W_\sigma}, \bar{\partial}^{W_\sigma})$. By construction we have a natural identification

$$\phi_{\sigma, m} : (W_\sigma)_{| \{m \times \mathbb{CP}^1\}} \xrightarrow{\sim} (\tilde{W}_\sigma)_{| \{m \times \mathbb{CP}^1\}} .$$

Indeed, both sides are canonically isomorphic to the vector bundle on \mathbb{CP}^1 corresponding to

$$(U_0 \otimes_\sigma \mathbb{C}) \otimes \mathcal{O}_{\mathbb{CP}^1}(1) \oplus (U_2 \otimes_\sigma \mathbb{C}) \otimes \mathcal{O}_{\mathbb{CP}^1} .$$

We use parallel transport along curves with constant projection to \mathbb{CP}^1 in order to extend this identification to an isomorphism

$$\phi_\sigma : W_\sigma \rightarrow \tilde{W}_\sigma$$

which is compatible with the partial connections ∇^{I, W_σ} and $\nabla^{I, \tilde{W}_\sigma}$. This works since by construction the isomorphism $\phi_{\sigma, m}$ maps the holonomy of $(\nabla^{I, W_\sigma})_{M \times \{z\}}$ to the holonomy of $(\nabla^{I, \tilde{W}_\sigma})_{M \times \{z\}}$ for all $z \in \mathbb{CP}^1$. Note that ϕ_σ is also compatible with the holomorphic structures $\bar{\partial}^{W_\sigma}, \bar{\partial}^{\tilde{W}_\sigma}$. We now use ϕ_σ in order to transport the geometry g^{W_σ} defined in 5.6.2 for the sequence (114) to a geometry $g^{\tilde{W}_\sigma}$.

We define the geometry $g^{\tilde{V}_1}$ such that it evaluates at all $\sigma \in X(\mathbb{C})$ to the geometry $g^{\tilde{W}_\sigma}$. We let $u_i : M \times X \rightarrow M \times X \times Y$ be the maps given by the points $[i : 1] \in \mathbb{P}_{\mathbb{Z}}^1(\mathbb{Z})$, $i = 0, 1$. By our normalization of the metric $h^{H_Y(1)}$ we have a natural isomorphism

$$u_1^*(\tilde{V}_1, g^{\tilde{V}_1}) \cong (V_1, g^{V_1}) .$$

We also consider the split variant $(\tilde{V}_1^{\text{split}}, g^{\tilde{V}_1^{\text{split}}})$ obtained by replacing the exact sequence (113) in the constructions above by

$$\mathcal{V}^{\text{split}} : 0 \rightarrow V_0 \rightarrow V_0 \oplus V_2 \rightarrow V_2 \rightarrow 0$$

and using the geometry induced by g^{V_0} and g^{V_2} on $V_0 \oplus V_2$. By construction we have natural isomorphisms

$$u_0^*(\tilde{V}_1^{\text{split}}, g^{\tilde{V}_1^{\text{split}}}) \cong u_0^*(\tilde{V}_1, g^{\tilde{V}_1}), \quad u_1^*(\tilde{V}_1^{\text{split}}, g^{\tilde{V}_1^{\text{split}}}) \cong (V_0, g^{V_0}) \oplus (V_2, g^{V_2}) .$$

We write

$$x := \widehat{\text{cyc1}}(\tilde{V}_1, g^{\tilde{V}_1}), \quad y := \widehat{\text{cyc1}}(\tilde{V}_1^{\text{split}}, g^{\tilde{V}_1^{\text{split}}}) .$$

Since the topological cycle map (74) is additive on short exact sequences we have $I(x) = I(y)$. Furthermore $\int_{II} R(y) = 0$ (cf. the argument in Subsection 5.6.2). By Proposition 5.8 we have

$$u_1^*(x) - u_1^*(y) = u_1^*(x - y) - u_0^*(x - y) = a\left(\int_{II} R(x)\right) = a\left(\int_{II} \omega(g^{\tilde{V}_1})\right).$$

In other words,

$$\widehat{\text{cyc1}}(V_0, g_0) - \widehat{\text{cyc1}}(V_1, g_1) + \widehat{\text{cyc1}}(V_2, g_2) = -a\left(\int_{II} \omega(g^{\tilde{V}_1})\right) .$$

Note that

$$\text{ev}_\sigma \int_{II} \omega(g^{\tilde{V}_1}) = \mathcal{U}(\mathcal{V}_\sigma, g^{\mathcal{V}_\sigma})$$

for all $\sigma \in X(\mathbb{C})$. In view of (117) this implies

$$-a\left(\int_{II} \omega(g^{\tilde{V}_1})\right) = a(\delta(\mathcal{V}, g^{\mathcal{V}}))$$

and thus Theorem 5.16. □

A Notations and technicalities

A.1 Categories

We list some categories appearing in the present paper.

name	symbol
sets	Set
simplicial sets	sSet
spectra (of simplicial sets)	Sp
abelian groups	Ab
chain complexes (of abelian groups)	Ch
standard finite ordered sets	Δ
1-category of categories	Cat

A.2 ∞ -categories and their localization

We use the language of ∞ -categories. A basic reference is [Lur09]. A quick overview is given in [Gro10]. For us, an ∞ -category is a simplicial set satisfying the inner lifting property [Lur09, 1.1.2.4]. Given a simplicial set S and an ∞ -category \mathbf{C} , we denote by $\mathbf{Fun}(S, \mathbf{C})$ the simplicial set of maps between the underlying simplicial sets. It is again an ∞ -category [Lur09, 1.2.7.3]. A functor between ∞ -categories is a morphism of simplicial sets. It is an equivalence of ∞ -categories if the morphism of simplicial sets is a categorical equivalence [Lur09, 1.1.5.14].

Let \mathbf{C} be some ∞ -category and W a collection of morphisms, i.e. 1-simplices, called weak equivalences. For every ∞ -category \mathbf{T} we let

$$\mathbf{Fun}_{W^{-1}}(\mathbf{C}, \mathbf{T}) \subseteq \mathbf{Fun}(\mathbf{C}, \mathbf{T})$$

be the full subcategory of functors which map the weak equivalences W to equivalences in \mathbf{T} . Then there exists an ∞ -category $\mathbf{C}[W^{-1}]$ together with a natural functor

$$\mathbf{C} \rightarrow \mathbf{C}[W^{-1}]$$

characterized by the universal property that for every ∞ -category \mathbf{T} the induced map on functors to \mathbf{T} factorizes over an equivalence

$$\mathbf{Fun}(\mathbf{C}[W^{-1}], \mathbf{T}) \xrightarrow{\sim} \mathbf{Fun}_{W^{-1}}(\mathbf{C}, \mathbf{T})$$

(cf. [Spi10, Lemma 5.1]).

Here we list the basic ∞ -categories and the choices of weak equivalences used in the present paper. We denote the nerve of a category \mathcal{C} by $\mathbf{N}(\mathcal{C})$. It will be considered as an ∞ -category.

name	symbol	W
categories	$\mathbf{N}(\mathbf{Cat})$	equivalences of categories
simplicial sets	$\mathbf{N}(\mathbf{sSet})$	weak homotopy equivalences
chain complexes of abelian groups	$\mathbf{N}(\mathbf{Ch})$	quasi-isomorphisms
spectra	$\mathbf{N}(\mathbf{Sp})$	stable equivalences

A.3 Monoids and groups

If \mathbf{C} is a symmetric monoidal ∞ -category, then have the ∞ -category $\mathbf{CommMon}(\mathbf{C})$ of commutative monoids in \mathbf{C} (called commutative algebra objects in [Lur, Definition 2.1.3.1]). In the special case of $\mathbf{N}(\mathbf{sSet})[W^{-1}]$ with the cartesian symmetric monoidal structure we can consider the full subcategory of grouplike monoids. The inclusion is part of an adjunction

$$\Omega B : \mathbf{CommMon}(\mathbf{N}(\mathbf{sSet})[W^{-1}]) \rightleftarrows \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}]) : \text{inclusion} ,$$

where ΩB is called the group completion.

We let $\mathbf{N}(\mathbf{Sp}^{\geq 0}) \subset \mathbf{N}(\mathbf{Sp})$ denote the full subcategory of connective spectra. Its weak equivalences are induced from $\mathbf{N}(\mathbf{Sp})$. The ∞ -loop space functor from spectra to pointed simplicial sets refines to a functor

$$\Omega^\infty : \mathbf{N}(\mathbf{Sp})[W^{-1}] \rightarrow \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}])$$

whose restriction to $\mathbf{N}(\mathbf{Sp}^{\geq 0})[W^{-1}]$ is an equivalence. We write \mathbf{sp} for the composition

$$\mathbf{sp} : \mathbf{CommGroup}(\mathbf{N}(\mathbf{sSet})[W^{-1}]) \xrightarrow{(\Omega^\infty)^{-1}} \mathbf{N}(\mathbf{Sp}^{\geq 0})[W^{-1}] \rightarrow \mathbf{N}(\mathbf{Sp})[W^{-1}] .$$

A.4 Mapping spaces

If \mathbf{C} is an ∞ -category and $x, y \in \mathbf{C}$ are objects, then we let

$$\mathbf{map}(x, y) \in \mathbf{N}(\mathbf{sSet})[W^{-1}]$$

denote the mapping space [Lur09, 1.2.2.1]. If \mathbf{C} is stable [Lur, 1.1.1.9] (as e.g. $\mathbf{N}(\mathbf{Sp})[W^{-1}]$ or $\mathbf{N}(\mathbf{Ch})[W^{-1}]$), then we have a mapping spectrum

$$\mathbf{Map}(x, y) \in \mathbf{N}(\mathbf{Sp})[W^{-1}]$$

which refines the mapping space in the sense that $\Omega^\infty \mathbf{Map}(x, y) \cong \mathbf{map}(x, y)$ (where we secretly forget the commutative group structure on the left hand side).

In the case that

$$x, y \in \mathbf{CommMon}(\mathbf{N}(\mathbf{sSet})[W^{-1}])$$

we write

$$\underline{\mathbf{Map}}(x, y) \in \mathbf{CommMon}(\mathbf{N}(\mathbf{sSet})[W^{-1}])$$

for the mapping space $\mathbf{map}(x, y)$ equipped with the additional structure of a commutative monoid. Details will always be indicated at the corresponding places.

A.5 The Eilenberg-MacLane correspondence

We consider $\mathbb{Z} \in \mathbf{N}(\mathbf{Ch})[W^{-1}]$ as a chain complex in degree 0 and let

$$H\mathbb{Z} := \mathbf{Map}(\mathbb{Z}, \mathbb{Z}) \in \mathbf{CommMon}(\mathbf{N}(\mathbf{Sp})[W^{-1}])$$

be its endomorphism spectrum with the additional structure of a commutative ring spectrum. We let $\mathbf{Mod}(H\mathbb{Z})$ denote the ∞ -category of $H\mathbb{Z}$ -modules which comes with a forgetful functor

$$\mathbf{Mod}(H\mathbb{Z}) \rightarrow \mathbf{N}(\mathbf{Sp})[W^{-1}].$$

Then there is a unique equivalence (the Eilenberg-MacLane functor) of stable ∞ -categories

$$H : \mathbf{N}(\mathbf{Ch})[W^{-1}] \xrightarrow{\sim} \mathbf{Mod}(H\mathbb{Z})$$

which sends \mathbb{Z} to $H\mathbb{Z}$. We refer to [BG, Subsection 6.8].

We need the following relation of the Eilenberg-MacLane correspondence with the group

completion:

$$\begin{array}{ccc}
\mathbf{N}(\mathbf{Ab}) & \xrightarrow{\quad} & \mathbf{CommMon}(\mathbf{N}(\mathbf{Cat})[W^{-1}]) \\
\downarrow S^0 & \searrow \text{const} & \downarrow \text{Noi} \\
& \mathbf{N}(s\mathbf{Ab}) & \xrightarrow{\quad} \mathbf{CommMon}(\mathbf{N}(s\mathbf{Set})[W^{-1}]) \\
& \downarrow & \downarrow \Omega B \\
& \mathbf{N}(s\mathbf{Ab})[W^{-1}] & \xrightarrow{\quad} \mathbf{CommGroup}(\mathbf{N}(s\mathbf{Set})[W^{-1}]) \\
& \downarrow \text{Dold-Kan} \cong & \downarrow \text{sp} \\
& \mathbf{N}(\mathbf{Ch}^{\leq 0})[W^{-1}] & \\
\swarrow \text{incl.} & \xrightarrow{H} & \downarrow \\
\mathbf{N}(\mathbf{Ch})[W^{-1}] & \xrightarrow{\quad} & \mathbf{N}(\mathbf{Sp})[W^{-1}]
\end{array} \quad (119)$$

A.6 Primitive cohomology classes

Here we provide some technical facts used in the proof of Proposition 3.8. We first establish the isomorphism (30).

Recall that

$$Y : (\mathbf{Sm}_{\mathbb{C}})^{\Delta^{op}} \rightarrow \mathbf{Sh}_{s\mathbf{Set}}(\mathbf{Sm}_{\mathbb{C}})$$

is the Yoneda embedding and $\kappa : \mathbf{Sh}_{s\mathbf{Set}}(\mathbf{Sm}_{\mathbb{C}}) \rightarrow \mathbf{Fun}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(s\mathbf{Set})[W^{-1}])$ is the canonical map. We denote the ∞ -categorical Yoneda image of the variety $BGL_q \in \mathbf{Sm}_{\mathbb{C}}$ by $Y_{\infty}(BGL_q) \in \mathbf{Fun}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(s\mathbf{Set})[W^{-1}])$. Then we have a canonical equivalence²

$$\kappa(Y(BGL_{\bullet})) \cong \text{colim}_{[q] \in \Delta^{op}} Y_{\infty}(BGL_q)$$

in $\mathbf{Fun}(\mathbf{S}_{\mathbb{C}}, \mathbf{N}(s\mathbf{Set})[W^{-1}])$. Using this and the Yoneda Lemma [Lur09, Lemma 5.1.5.2] we get the equivalences

$$\begin{aligned}
\text{map}(\kappa(Y(BGL_{\bullet})), \Omega^{\infty} H(\mathbf{DR}(p))) &\cong \lim_{[q] \in \Delta} \text{map}(Y_{\infty}(BGL_q), \Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p))) \\
&\cong \lim_{[q] \in \Delta} \Omega^{\infty} H(\mathbf{DR}_{\mathbb{C}}(p)(BGL_q))
\end{aligned}$$

The functors Ω^{∞} and H commute with limits. Since moreover $\lim_{[q] \in \Delta} \mathbf{DR}_{\mathbb{C}}(p)(BGL_q) \cong \text{tot} \mathbf{DR}_{\mathbb{C}}(p)(BGL_{\bullet})$ we obtain the equivalence

$$\text{map}(\kappa(Y(BGL_{\bullet})), \Omega^{\infty} H(\mathbf{DR}(p))) \cong \Omega^{\infty} H(\text{tot}(\mathbf{DR}_{\mathbb{C}}(p)(BGL_{\bullet}))).$$

Since for any complex $A \in \mathbf{N}(\mathbf{Ch})[W^{-1}]$ we have $\pi_k(H(A)) \cong H^{-k}(A)$ we get for any $k \geq 0$ a natural isomorphism

$$\pi_k(\text{map}(\kappa(Y(BGL_{\bullet})), \Omega^{\infty} H(\mathbf{DR}(p)))) \cong H^{-k}(\text{tot}(\mathbf{DR}_{\mathbb{C}}(p)(BGL_{\bullet})))$$

²If S is a bisimplicial set there is a natural equivalence $\text{diag } S \cong \text{colim}_{\Delta^{op}} S$ in $\mathbf{N}(s\mathbf{Set})[W^{-1}]$. We apply this to a bisimplicial set which is constant in one direction.

which is exactly (30).

The direct sum makes BGL_\bullet a monoid in $\mathbf{Sm}_\mathbb{C}^{\Delta^{op}}$. We denote by $\mu, \text{pr}_1, \text{pr}_2 : BGL_\bullet \times BGL_\bullet \rightarrow BGL_\bullet$ the monoid structure and the projections, respectively. The subspace of primitive elements

$$H^*(\text{tot}(\mathbf{DR}_\mathbb{C}(p)(BGL_\bullet)))^{prim} \subseteq H^*(\text{tot}(\mathbf{DR}_\mathbb{C}(p)(BGL_\bullet))) \quad (120)$$

is by definition the set of elements x satisfying

$$\mu^*(x) = \text{pr}_1^*x + \text{pr}_2^*x.$$

It follows from the above and from the definition of $\text{Prim}(M, \mathbf{DR}_\mathbb{C}(p))$ in 3.7 that we have an injection

$$\text{Prim}(M, \mathbf{DR}_\mathbb{C}(p)) \hookrightarrow H^0(\text{tot}(\mathbf{DR}_\mathbb{C}(p)(BGL_\bullet)))^{prim}. \quad (121)$$

A.7 Sheaves

We now consider a site S , i.e. a category with a topology determined by a collection of coverings $(U \rightarrow M)$, and take $\mathbf{S} := \mathbf{N}(S^{op})$. For the following we refer to [Lur09, 6.2.2.6-6.2.2.7]. We assume that the target \mathbf{C} is a presentable ∞ -category.

Definition A.1. *We let*

$$\mathbf{Fun}^{desc}(\mathbf{S}, \mathbf{C}) \subseteq \mathbf{Fun}(\mathbf{S}, \mathbf{C})$$

denote the full subcategory of sheaves, i.e. objects $F \in \mathbf{Fun}(\mathbf{S}, \mathbf{C})$ which satisfy the descent condition that

$$F(M) \xrightarrow{\sim} \lim_{\Delta} F(U_\bullet)$$

is an equivalence in \mathbf{C} for all open coverings $U \rightarrow M$, where $U_\bullet \in S^{\Delta^{op}}$ denotes the associated Čech nerve.

We have an adjunction

$$L : \mathbf{Fun}(\mathbf{S}, \mathbf{C}) \rightleftarrows \mathbf{Fun}^{desc}(\mathbf{S}, \mathbf{C}) : \text{inclusion}, \quad (122)$$

where L is called the sheafification.

Here is a list of sites used in the present paper.

site	∞ -category	name	coverings
Mf	\mathbf{S}_{Mf}	manifolds	open coverings
Sm $_\mathbb{C}$	$\mathbf{S}_\mathbb{C}$	smooth varieties over \mathbb{C}	Zariski open coverings
Reg $_\mathbb{Z}$	$\mathbf{S}_\mathbb{Z}$	regular, separated schemes of finite type over \mathbb{Z}	Zariski open coverings
Mf \times Reg $_\mathbb{C}$	$\mathbf{S}_{Mf, \mathbb{C}}$	the product	induced
Mf \times Reg $_\mathbb{Z}$	$\mathbf{S}_{Mf, \mathbb{Z}}$	the product	induced

A.8 Homotopy invariance

Let $I := [0, 1] \in \mathbf{Mf}$ be the unit interval and $\mathbf{S} \in \{\mathbf{S}_{Mf}, \mathbf{S}_{Mf, \mathbb{C}}, \mathbf{S}_{Mf, \mathbb{Z}}\}$. For any ∞ -category \mathbf{C} the pull-back along the multiplication by I gives an endofunctor $\mathcal{I} : \mathbf{Fun}(\mathbf{S}, \mathbf{C}) \rightarrow \mathbf{Fun}(\mathbf{S}, \mathbf{C})$. Moreover, the projection along I gives a transformation $\text{id} \rightarrow \mathcal{I}$.

Definition A.2. *We call a functor $X \in \mathbf{Fun}(\mathbf{S}, \mathbf{C})$ homotopy invariant if the natural morphism $X \rightarrow \mathcal{I}(X)$ is an equivalence.*

Let $\Delta^\bullet \in \mathbf{Mf}^\Delta$ be the cosimplicial manifold given by the standard simplices. It gives a functor $\mathbf{S} \times \mathbb{N}(\Delta^{op}) \rightarrow \mathbf{S}$ induced by $(M, [q]) \mapsto M \times \Delta^q$. Pull-back along this functor defines the functor

$$\mathbf{s} : \mathbf{Fun}(\mathbf{S}, \mathbf{C}) \rightarrow \mathbf{Fun}(\mathbf{S} \times \mathbb{N}(\Delta^{op}), \mathbf{C}) . \quad (123)$$

We define the endofunctor

$$\bar{\mathbf{s}} := \text{colim}_{\mathbb{N}(\Delta^{op})} \circ \mathbf{s} : \mathbf{Fun}(\mathbf{S}, \mathbf{C}) \rightarrow \mathbf{Fun}(\mathbf{S}, \mathbf{C}) . \quad (124)$$

The projection $\Delta^\bullet \rightarrow *$ to the constant cosimplicial manifold given by the point induces the transformation

$$\text{id} \rightarrow \bar{\mathbf{s}} . \quad (125)$$

It is a simple exercise using the contractibility of the simplices to show the following.

Lemma A.3. *If $X \in \mathbf{Fun}(\mathbf{S}, \mathbf{C})$ is homotopy invariant, then $X \rightarrow \bar{\mathbf{s}}(X)$ is an equivalence.*

A.9 Constant sheaves

Let \mathbf{C} be a presentable ∞ -category. The projection $p : \mathbf{S}_{Mf} \rightarrow *$ induces a functor

$$p^* : \mathbf{C} \rightarrow \mathbf{Fun}(\mathbf{S}_{Mf}, \mathbf{C}) .$$

Furthermore, if we let $e : \{*\} \rightarrow \mathbf{S}_{Mf}$ be the inclusion and

$$e^* : \mathbf{Fun}(\mathbf{S}_{Mf}, \mathbf{C}) \rightarrow \mathbf{C}$$

be the evaluation at $*$, then for every $X \in \mathbf{Fun}(\mathbf{S}_{Mf}, \mathbf{C})$ we have a natural map $p^*e^*X \rightarrow X$. If X is a sheaf then this morphism extends naturally to a morphism $L(p^*e^*X) \rightarrow X$, where $L : \mathbf{Fun}(\mathbf{S}_{Mf}, \mathbf{C}) \rightarrow \mathbf{Fun}^{desc}(\mathbf{S}_{Mf}, \mathbf{C})$ denotes the sheafification functor (122).

Definition A.4. *We say that $X \in \mathbf{Fun}^{desc}(\mathbf{S}_{Mf}, \mathbf{C})$ is constant if the natural morphism $L(p^*e^*X) \rightarrow X$ is an equivalence.*

We let

$$\mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf}, \mathbf{C}) \subseteq \mathbf{Fun}^{desc}(\mathbf{S}_{Mf}, \mathbf{C})$$

be the full subcategory of constant sheaves. Furthermore, we define

$$\mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf, \mathbb{Z}}, \mathbf{C}) := \mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf}, \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{Z}}, \mathbf{C})) , \quad (126)$$

$$\mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf, \mathbb{C}}, \mathbf{C}) := \mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf}, \mathbf{Fun}^{desc}(\mathbf{S}_{\mathbb{C}}, \mathbf{C}))$$

i.e. the constancy condition only refers to the manifold direction.

Using the fact that manifolds admit good open coverings, i.e. open coverings such that all components of multiple intersections are contractible, one can show:

Lemma A.5. *If $X \in \mathbf{Fun}^{desc}(\mathbf{S}_{Mf}, \mathbf{C})$ is homotopy invariant (see A.2), then it is constant.*

A.10 Sm - smooth function objects

We have functors $\mathbf{sing}_{\infty} \subseteq \mathbf{sing} : \mathbf{Mf} \rightarrow \mathbf{sSet}$ which associate to a manifold its singular complex and its subcomplex of smooth singular simplices. Since manifolds are locally contractible we have

$$\mathbf{sing} \in \mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf}, \mathbf{N}(\mathbf{sSet})[W^{-1}]^{op}) .$$

Moreover, the natural inclusion

$$\mathbf{sing}_{\infty} \hookrightarrow \mathbf{sing} \tag{127}$$

is an equivalence in $\mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf}, \mathbf{N}(\mathbf{sSet})[W^{-1}]^{op})$.

Let \mathbf{C} be a presentable ∞ -category. Then it is cotensored over $\mathbf{N}(\mathbf{sSet})[W^{-1}]$. We denote the cotensor by

$$\cot : \mathbf{N}(\mathbf{sSet})[W^{-1}]^{op} \times \mathbf{C} \rightarrow \mathbf{C} .$$

Using the cotensor structure we can define the functor

$$\mathbf{Sm} := \cot \circ (\mathbf{sing} \times \mathrm{id}) : \mathbf{C} \rightarrow \mathbf{Fun}^{desc, const}(\mathbf{S}_{Mf}, \mathbf{C}) .$$

We say that $\mathbf{Sm}(U)$ is the smooth function object associated to $U \in \mathbf{C}$.

Sometimes we use the equivalence

$$\mathbf{Sm} \cong \cot \circ (\mathbf{sing}_{\infty} \times \mathrm{id}) \tag{128}$$

induced by (127).

Corollary A.6. *For every $X \in \mathbf{C}$ the smooth function object $\mathbf{Sm}(X)$ is homotopy invariant, and hence constant.*

Let $\mathbf{S} \in \{\mathbf{S}_{Mf}, \mathbf{S}_{Mf, \mathbb{C}}, \mathbf{S}_{Mf, \mathbb{Z}}\}$ and p be the projection from \mathbf{S} to the point, respectively to the second factor, which belongs to $\{*, \mathbf{S}_{\mathbb{C}}, \mathbf{S}_{\mathbb{Z}}\}$, correspondingly. Similarly let e denote the inclusion of the point respectively the second factor into \mathbf{S} given by $*$ in \mathbf{S}_{Mf} .

Lemma A.7. *If $X \in \mathbf{Fun}^{desc, const}(\mathbf{S}, \mathbf{C})$, then we have natural equivalences*

$$X \xleftarrow{\sim} L(p^* e^* X) \cong L(p^* e^* (\mathbf{Sm}(e^* X))) \xrightarrow{\sim} \mathbf{Sm}(e^* X) .$$

Proof. This follows from the constancy of X and $\mathbf{Sm}(e^* X)$. □

Let $E \in \mathbf{N}(\mathbf{Sp})[W^{-1}]$ be a spectrum. It represents a cohomology theory E^* . For every $i \in \mathbb{Z}$ and $M \in \mathbf{Mf}$ we have

$$\pi_i(\mathbf{Sm}(E)(M)) \cong E^{-i}(M) . \tag{129}$$

A.11 Complexes and cones

Using the convention $C_{-i} = C^i$ a chain complex (C, ∂_C) will either be indexed cohomologically,

$$\dots \rightarrow C^i \xrightarrow{\partial_C} C^{i+1} \rightarrow \dots ,$$

or homologically,

$$\dots \rightarrow C_{-i} \xrightarrow{\partial_C} C_{-i-1} \rightarrow \dots .$$

For an integer k we define the shifted complex $(C[k], \partial_{C[k]})$ by $C[k]^i = C^{i+k}$ or equivalently $C[k]_i = C_{i-k}$ with differential $\partial_{C[k]} = (-1)^k \partial_C$.

We fix the conventions for the cone of a map of chain complexes $\phi : C \rightarrow D$ as follows. We set

$$\mathbf{Cone}(\phi)^i := C^{i+1} \oplus D^i .$$

The differential of $\mathbf{Cone}(\phi)$ is given by

$$\partial(c, d) := (-\partial_C(c), \partial_D(d) - \phi(c)) .$$

The cone fits into a fibre sequence

$$\dots \rightarrow \mathbf{Cone}(\phi)[-1] \rightarrow C \xrightarrow{\phi} D \rightarrow \mathbf{Cone}(\phi) \rightarrow \dots$$

in $\mathbf{N}(\mathbf{Ch})[W^{-1}]$.

A.12 Variants of differential forms

1. $A(M \times X)$ denotes the smooth complex-valued differential forms on the smooth manifold $M \times X$, where X is a complex manifold.
2. $A_{\mathbb{R}}(M \times X)$ denotes the subcomplex of real forms.
3. For $X \in \mathbf{Sm}_{\mathbb{C}}$ we denote by $A_{\log, Mf}(M \times X)$, $A_{\log, Mf, \mathbb{R}}(M \times X)$ the complex of forms with logarithmic singularities at the boundary of X and its subcomplex of real-valued forms introduced in (46). They carry the increasing weight filtration \mathcal{W}_* . In addition, $A_{\log, Mf}(M \times X)$ carries the decreasing Hodge filtration \mathcal{F}^* .
4. $\mathbf{DR}_{Mf, \mathbb{C}}(M \times X)$ and $\mathbf{DR}_{Mf, \mathbb{Z}}(M \times X)$ denote the cones introduced in Definitions 4.3 and 4.4. They are products over the components $\mathbf{DR}_{Mf, \mathbb{C}}(p)(M \times X)$ or $\mathbf{DR}_{Mf, \mathbb{Z}}(p)(M \times X)$, respectively.
5. $\mathbf{DR}_{\mathbb{C}}(X) := \mathbf{DR}_{Mf, \mathbb{C}}(* \times X)$, $\mathbf{DR}_{\mathbb{Z}}(X) := \mathbf{DR}_{Mf, \mathbb{Z}}(* \times X)$

References

- [Beĭ84] A. A. Beĭlinson, *Higher regulators and values of L -functions*, Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238. MR 760999 (86h:11103)
- [Beĭ86] ———, *Notes on absolute Hodge cohomology*, Applications of algebraic K -theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 35–68. MR 862628 (87m:14019)
- [BG] Ulrich Bunke and David Gepner, *Differential function spectra and differential Becker-Gottlieb transfer with application to differential algebraic K -theory*, in preparation.
- [BL95] Jean-Michel Bismut and John Lott, *Flat vector bundles, direct images and higher real analytic torsion*, J. Amer. Math. Soc. **8** (1995), no. 2, 291–363. MR 1303026 (96g:58202)
- [Bor74] Armand Borel, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 235–272 (1975). MR 0387496 (52 #8338)
- [Bor77] ———, *Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), no. 4, 613–636. MR 0506168 (58 #22016)
- [BS10] Ulrich Bunke and Thomas Schick, *Uniqueness of smooth extensions of generalized cohomology theories*, J. Topol. **3** (2010), no. 1, 110–156. MR 2608479 (2011e:55011)
- [Bun12] Ulrich Bunke, *Differential cohomology*, Course notes, Regensburg, 2012, arXiv:1208.3961, 2012.
- [Bur94] José Ignacio Burgos, *A C^∞ logarithmic Dolbeault complex*, Compositio Math. **92** (1994), no. 1, 61–86. MR 1275721 (95g:32056)
- [BW98] Jose Ignacio Burgos and Steve Wang, *Higher Bott-Chern forms and Beilinson’s regulator*, Invent. Math. **132** (1998), no. 2, 261–305. MR 1621424 (99j:14008)
- [Che79] Jeff Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. (2) **109** (1979), no. 2, 259–322. MR 528965 (80j:58065a)
- [CS85] Jeff Cheeger and James Simons, *Differential characters and geometric invariants*, Geometry and topology (College Park, Md., 1983/84), Lecture Notes in Math., vol. 1167, Springer, Berlin, 1985, pp. 50–80. MR 827262 (87g:53059)

- [Del71] Pierre Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR 0498551 (58 #16653a)
- [Del74] ———, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. (1974), no. 44, 5–77. MR 0498552 (58 #16653b)
- [Del87] P. Deligne, *Le déterminant de la cohomologie*, Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), Contemp. Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 93–177. MR 902592 (89b:32038)
- [DWW03] W. Dwyer, M. Weiss, and B. Williams, *A parametrized index theorem for the algebraic K-theory Euler class*, Acta Math. **190** (2003), no. 1, 1–104. MR 1982793 (2004d:19004)
- [Gro10] Moritz Groth, *A short course on ∞ -categories*, preprint available from the author’s home page <http://www.math.ru.nl/~mgroth/>, July 2010.
- [GS90] Henri Gillet and Christophe Soulé, *Characteristic classes for algebraic vector bundles with Hermitian metric. I*, Ann. of Math. (2) **131** (1990), no. 1, 163–203. MR 1038362 (91m:14032a)
- [Hir64] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; *ibid.* (2) **79** (1964), 205–326. MR 0199184 (33 #7333)
- [Kar83] Max Karoubi, *Homologie cyclique et régulateurs en K-théorie algébrique*, C. R. Acad. Sci. Paris Sér. I Math. **297** (1983), no. 10, 557–560. MR 735692 (85f:18006)
- [Kar87] ———, *Homologie cyclique et K-théorie*, Astérisque (1987), no. 149, 147. MR 913964 (89c:18019)
- [Lot00] John Lott, *Secondary analytic indices*, Regulators in analysis, geometry and number theory, Progr. Math., vol. 171, Birkhäuser Boston, Boston, MA, 2000, pp. 231–293. MR 1724894 (2001c:58024)
- [Lur] Jacob Lurie, *Higher algebra*, Unpublished manuscript available from the author’s homepage <http://www.math.harvard.edu/~lurie/>.
- [Lur09] ———, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)
- [Mül78] Werner Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. **28** (1978), no. 3, 233–305. MR 498252 (80j:58065b)
- [Nag62] Masayoshi Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2** (1962), 1–10. MR 0142549 (26 #118)

- [Qui73] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR 0338129 (49 #2895)
- [Sch12] J. Scholbach, *Arakelov motivic cohomology II*, arXiv:1205.3890, May 2012.
- [Sou92] C. Soulé, *Lectures on Arakelov geometry*, Cambridge Studies in Advanced Mathematics, vol. 33, Cambridge University Press, Cambridge, 1992, With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer. MR 1208731 (94e:14031)
- [Spi10] M. Spitzweck, *Another viewpoint on J-spaces*, arXiv:1012.1264v2, December 2010.
- [Tak05] Yuichiro Takeda, *Higher arithmetic K-theory*, Publ. Res. Inst. Math. Sci. **41** (2005), no. 3, 599–681. MR 2153537 (2006i:14022)
- [Tam12] Georg Tamme, *Karoubi’s relative Chern character and Beilinson’s regulator*, Ann. Sci. Éc. Norm. Supér. (4) **45** (2012).
- [TT90] R. W. Thomason and Thomas Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435. MR 1106918 (92f:19001)
- [Wei] Charles A. Weibel, *Algebraic K-theory*, A graduate textbook in progress available from the author’s home page <http://www.math.rutgers.edu/~weibel/Kbook.html>.